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Averaged equations for inviscid disperse two-phase flow†

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Averaged equations governing the motion of equal rigid spheres suspended in a potential flow are derived from the equation for the probability distribution. A distinctive feature of this work is the derivation of the disperse-phase momentum equation by averaging the particle equation of motion directly, rather than the microscopic equation for the particle material. This approach is more flexible than the usual one and leads to a simpler and more fundamental description of the particle phase. The model is closed in a systematic way (i.e. with no *ad hoc* assumptions) in the dilute limit and in the linear limit. One of the closure quantities is related to the difference between the gradient of the average pressure and the average pressure gradient, a well-known problem in the widely used two-fluid engineering models. The present result for this quantity leads to the introduction of a modified added mass coefficient (related to Wallis's 'exertia') that remains very nearly constant with changes in the volume fraction and densities of the phases. Statistics of this coefficient are provided and exhibit a rather strong variability of up to 20% among different numerical simulations. A detailed comparison of the present results with those of other investigators is given in §10.

As a further illustration of the flexibility of the techniques developed in the paper, in Appendix C they are applied to the calculation of the so-called 'particle stress' tensor. This derivation is considerably simpler than others available in the literature.

1. Introduction

This paper presents a method for the derivation of averaged equations for disperse two-phase systems. The method is systematic (meaning that no *ad hoc* closure relations are required, at least at the lowest non-trivial order) and general, in that it may be applied to a variety of thermo-fluid and solid mechanics situations. Here it is implemented for the case of rigid spheres suspended in a potential flow. In a related paper (Zhang & Prosperetti 1994), the case of spheres with a variable radius is studied and a summary of other applications to heat conduction and convection, Stokes flow, and thermocapillary processes is given in Zhang (1993) and Prosperetti & Zhang (1993). Our results shed light on many others available in the literature and achieve some unification. A detailed discussion and comparison with the work of Biesheuvel, Drew, Lahey, Wallis, Sangani, and others is given in §10.

Although our point of departure is the equation satisfied by the particle distribution function (Biesheuvel & Spoelstra 1989; Biesheuvel & Gorissen 1990; Koch 1990; Sangani & Didwania 1993*a*), we use ensemble averaging over the individual phases

† With Appendix C by H. F. Bulthuis.

rather than over the entire mixture as is more customary (see e.g. Batchelor 1970, 1972, 1974). For the disperse phase we average the particle momentum equation directly, rather than the momentum equation for the particle material as done by others (see e.g. Hinch 1977). This procedure offers several advantages as discussed in §3. In particular, the disperse-phase momentum equation found in this way involves the gradient of the continuous rather than the disperse-phase pressure. Our results are couched in the framework of the standard two-fluid model of engineering multiphase flow, but their equivalence with other approaches, e.g. the one in which a mixture momentum and a disperse-phase impulse equation are formulated, is obvious and is discussed in §10.

The closure of the model involves, among other quantities, the average continuous-phase pressure gradient, whose relation with the gradient of the average pressure is a well-known issue in multiphase flow theory (see e.g. Drew 1983; Prosperetti & Jones 1984). This point is of particular importance because it is in relating these two quantities that many of the interphase momentum transfer effects – in particular, added mass – arise. Such closure terms are calculated explicitly for dilute suspensions by analytical means and numerically for the linear problem at finite volume fractions. The average pressure gradient is related to the gradient of the average pressure by the introduction of a coefficient that, in this case, is related to the added mass coefficients studied by Zuber (1964) and many others (see e.g. van Wijngaarden 1976; Kok 1989; Biesheuvel & Spoelstra 1989; Sangani, Zhang & Prosperetti 1991), and to the 'exertia' of Wallis (1991*a*, *b*). This new coefficient has, however, a much smaller range of variation in its dependence on volume fractions and densities and its use may therefore offer some practical advantages.

Section 2 develops the basic mathematical tools, in §3 the general form of the averaged equations of motion is derived, and the 'small-particle' case is discussed in §4. The dilute-limit results are presented in §5 with details of the calculation given in Appendix B. Section 6 is devoted to a discussion of small-amplitude oscillatory motion at finite volume fraction. Section 7 contains some details relevant for the numerical simulation of this case and the results are presented in §§8 and 9. A general discussion and comparison is given in §10.

2. Preliminaries

We consider N identical spherical homogeneous particles in an infinite, unbounded, inviscid, incompressible fluid that constitutes the continuous phase. A specific configuration $\mathscr{C}^N$ of the system is specified by a set of position vectors $\boldsymbol{y}^\alpha$ and velocity vectors $\boldsymbol{w}^\alpha$ for $\alpha = 1, 2, \ldots, N$. With suitable initial conditions, dynamical equations for the particles and the continuous phase, and conditions 'at infinity' for the latter, it is then possible in principle to uniquely calculate the evolution of the system in time. Since the particles are indistinguishable, which particle occupies which position is irrelevant.

We consider an ensemble of realizations and denote by $P(N;t) \equiv P(1,2,\ldots,N;t)$ the probability of a specific configuration $\mathscr{C}^N$. In view of the identity of the particles the appropriate normalization is

$$N! = \int \mathrm{d}\mathscr{C}^N P(N;t)$$

$$\equiv \int \mathrm{d}^3y^{(1)} \int \mathrm{d}^3w^{(1)} \int \mathrm{d}^3y^{(2)} \int \mathrm{d}^3w^{(2)} \cdots \int \mathrm{d}^3y^{(N)} \int \mathrm{d}^3w^{(N)}\, P(1,2,\ldots,N;t). \quad (2.1)$$

The equation expressing the conservation of the number of realizations constituting the ensemble under consideration is

$$\frac{\partial P}{\partial t}+\sum_{\alpha=1}^{N}\left[\nabla_\alpha\cdot(\boldsymbol{w}^\alpha P)+\Delta_\alpha\cdot(\dot{\boldsymbol{w}}^\alpha P)\right]=0, \tag{2.2}$$

where we have introduced the abbreviated notation

$$\nabla_\alpha\equiv\nabla_{\boldsymbol{y}^\alpha},\quad \Delta_\alpha\equiv\nabla_{\boldsymbol{w}^\alpha}. \tag{2.3}$$

These derivatives are of course to be understood in the generalized sense if P is not smooth. Equation (2.2) is to be integrated subject to a suitable initial condition and to the condition of vanishing at infinity both in velocity and in physical space.

The reduced probability distribution $P(K;t)$ in which the configuration of K particles is specified is obtained from $P(N;t)$ by integration,

$$P(K;t)=\frac{1}{(N-K)!}\int P(N,t)\,\mathrm{d}\mathscr{C}^{N-K}, \tag{2.4}$$

and satisfies the normalization condition

$$\int P(K;t)\,\mathrm{d}\mathscr{C}^{K}=\frac{N!}{(N-K)!}. \tag{2.5}$$

Let $\chi_C(\boldsymbol{x};N)$ be the indicator function of the continuous phase in the presence of the configuration $\mathscr{C}^N$, i.e. $\chi_C=1$ when, given this configuration, $\boldsymbol{x}$ is in the continuous phase, and $\chi_C=0$ otherwise (see e.g. Drew 1983; Joseph & Lundgren 1990). Similarly, χ_D will denote the indicator function of the disperse phase. Since the particle boundaries have zero measure, $\chi_C+\chi_D=1$. The volume fractions β_C of the continuous and β_D of the disperse phase are defined in terms of these functions by

$$\beta_{C,D}(\boldsymbol{x},t)=\frac{1}{N!}\int\mathrm{d}\mathscr{C}^{N}\,P(N;t)\,\chi_{C,D}(\boldsymbol{x};N), \tag{2.6}$$

from which, by (2.1),

$$\beta_C+\beta_D=1. \tag{2.7}$$

It is important to realize that the volume fractions depend on time only through the time dependence of the probability $P(N;t)$. The indicator function is a purely geometrical entity only depending on the configuration and on $\boldsymbol{x}$. An alternative description would be possible in which the probability function refers to the initial condition and is independent of time, with all the quantities such as the indicator functions depending on time following the evolution of the system. The difference between the two points of view is conceptually similar to the Eulerian and Lagrangian descriptions in fluid mechanics. Just as in that case, the second description requires the introduction of the mapping between the initial and the current configuration and is therefore less convenient than the Eulerian-like description that we adopt.

Similarly to (2.6), we may define conditional volume fractions $\beta^K_{C,D}(\boldsymbol{x},t\,|\,K)$ by

$$\begin{aligned}\beta^K_{C,D}(\boldsymbol{x},t\,|\,K)&=\frac{1}{(N-K)!\,P(K;t)}\int\mathrm{d}\mathscr{C}^{N-K}\,P(N;t)\,\chi_{C,D}(\boldsymbol{x};N)\\&=\frac{1}{(N-K)!}\int\mathrm{d}\mathscr{C}^{N-K}\,P(N-K\,|\,K;t)\,\chi_{C,D}(\boldsymbol{x};N),\end{aligned} \tag{2.8}$$

where the conditional probability $P(N-K\,|\,K;t)$, defined by

$$P(N;t)=P(N-K\,|\,K;t)\,P(K;t), \tag{2.9}$$

has been introduced. From (2.1) and (2.5) one deduces

$$\int \mathrm{d}\mathscr{C}^{N-K} P(N-K|K;t) = (N-K)! \tag{2.10}$$

It is easy to show that the conditional volume fractions satisfy the same relation (2.7) as the unconditional ones.

For a configuration of equal spherical particles with radius a an explicit representation of the indicator function of the disperse phase is

$$\chi_D(\boldsymbol{x};N) = 1-\chi_C = \sum_{\alpha=1}^{N} H(a-|\boldsymbol{x}-\boldsymbol{y}^\alpha|), \tag{2.11}$$

with H the Heaviside distribution. In writing this equation we differ from the standard approach of kinetic theory (see e.g. Irving & Kirkwood 1950; Cercignani 1988), in which the spatial extent of the particles is ignored and the indicator function is written as

$$\chi_D(\boldsymbol{x};N) = v\sum_{\alpha=1}^{N} \delta(\boldsymbol{x}-\boldsymbol{y}^\alpha), \tag{2.12}$$

where v is the particle volume. This form of the characteristic function has recently been used by several researchers (Biesheuvel & Spoelstra 1989; Biesheuvel & Gorissen 1990; Koch 1990; Sangani & Didwania 1993*a*). On a technical level, it is the difference between (2.11) and (2.12) that renders the developments in this and the following section necessary.

With (2.11) we find

$$\begin{aligned}\beta_D(\boldsymbol{x},t) &= \frac{1}{N!}\sum_{\alpha=1}^{N}\int \mathrm{d}\mathscr{C}^N H(a-|\boldsymbol{x}-\boldsymbol{y}^\alpha|)\,P(N;t)\\ &= \frac{1}{(N-1)!}\int \mathrm{d}\mathscr{C}^1 H(a-|\boldsymbol{x}-\boldsymbol{y}^{(1)}|)\int \mathrm{d}\mathscr{C}^{N-1} P(N;t)\\ &= \int_{|\boldsymbol{x}-\boldsymbol{y}|\leqslant a} \mathrm{d}^3y \int \mathrm{d}^3w P(1;t),\end{aligned} \tag{2.13}$$

where $P(1;t) \equiv P(\boldsymbol{y},\boldsymbol{w};t)$. The step from the first to the second line is justified by the fact that the particles are all identical so that each of them gives the same contribution to the sum. The second step follows from the definition (2.4) of reduced probability distribution. By a similar calculation,

$$\beta_D^1(\boldsymbol{x},t\,|\,\boldsymbol{y},\boldsymbol{w}) = H(a-|\boldsymbol{x}-\boldsymbol{y}|) + \int_{|\boldsymbol{x}-\boldsymbol{y}'|\leqslant a} \mathrm{d}^3y' \int \mathrm{d}^3w' \frac{P(\boldsymbol{y},\boldsymbol{w},\boldsymbol{y}',\boldsymbol{w}';t)}{P(\boldsymbol{y},\boldsymbol{w};t)}. \tag{2.14}$$

We also define a local particle number density n by

$$\int \mathrm{d}^3w P(1;t) = n(\boldsymbol{y},t). \tag{2.15}$$

From the explicit representation (2.11) one finds the following expression for the space derivative of the continuous-phase indicator function:

$$\boldsymbol{\nabla}\chi_C = \sum_{\alpha=1}^{N} \delta(a-|\boldsymbol{x}-\boldsymbol{y}^\alpha|)\frac{\boldsymbol{x}-\boldsymbol{y}^\alpha}{|\boldsymbol{x}-\boldsymbol{y}^\alpha|}, \tag{2.16}$$

from which the gradient of the continuous phase volume fraction readily follows:

$$\nabla\beta_C = \frac{1}{N!}\sum_{\alpha=1}^{N}\int_{|x-y^\alpha|=a} \mathrm{d}S^{(\alpha)}\,\boldsymbol{n}^{(\alpha)}\int \mathrm{d}^3 w^{(\alpha)}\int \mathrm{d}\mathscr{C}^{N-1}\,P(N;t)$$

$$= \int_{|x-y|=a} \mathrm{d}S_y\,\boldsymbol{n}\int \mathrm{d}^3 w P(1;t) = \int_{|x-y|=a} \mathrm{d}S_y\,\boldsymbol{n}n(\boldsymbol{y},t), \tag{2.17}$$

where $\boldsymbol{n}$ denotes the unit normal oriented outward from the particle. The time derivative of β_C can be calculated by using (2.2) as follows:

$$\frac{\partial\beta_C}{\partial t} = \frac{1}{N!}\int \mathrm{d}\mathscr{C}^N \frac{\partial P(N;t)}{\partial t}\chi_C(\boldsymbol{x};N)$$

$$= -\frac{1}{N!}\int \mathrm{d}\mathscr{C}^N \chi_C \sum_{\alpha=1}^{N}[\nabla_\alpha\cdot(\boldsymbol{w}^\alpha P)+\Delta_\alpha\cdot(\dot{\boldsymbol{w}}^\alpha P)]$$

$$= -\frac{1}{N!}\int \mathrm{d}\mathscr{C}^N \sum_{\alpha=1}^{N}[\nabla_\alpha\cdot(\boldsymbol{w}^\alpha P\chi_C)+\Delta_\alpha\cdot(\dot{\boldsymbol{w}}^\alpha P\chi_C)]+\frac{1}{N!}\int \mathrm{d}\mathscr{C}^N P\sum_{\alpha=1}^{N}\boldsymbol{w}^\alpha\cdot\nabla_\alpha\chi_C. \tag{2.18}$$

Since P vanishes at infinity, the first integral vanishes. The terms $\nabla_\alpha\chi_C$ in the last sum can be calculated as in (2.16). Again in view of the identity of the particles, proceeding as in the derivation of (2.17), one then finds

$$\frac{\partial\beta_C}{\partial t} = -\int_{|x-y|=a} \mathrm{d}S_y\int \mathrm{d}^3 w\boldsymbol{n}\cdot\boldsymbol{w}\,P(1;t). \tag{2.19}$$

2.1. *Continuous-phase averages*

Let now $f_C(\boldsymbol{x},t;N)$ be any flow quantity pertaining to the continuous phase at position $\boldsymbol{x}$ and time t in the presence of the configuration $\mathscr{C}^N$ of the particles. To clarify the meaning of the time dependence explicitly indicated in f_C it is important to make the following remark. If the continuous phase moves only in response to the motion of the particles, in the framework of potential flow, any flow quantity is completely known when the position and velocity of the particles – i.e. $\mathscr{C}^N$ – are assigned. In this case f_C would not depend explicitly on time but only on $\boldsymbol{x}$ and $\mathscr{C}^N$. The time dependence is only necessary to account for the presence of external agents promoting flow other than the particles, e.g. moving boundaries or time-dependent forces.

The ensemble average of f_C is defined by averaging over all the configurations such that the point $\boldsymbol{x}$ is in the continuous phase,

$$\langle f_C\rangle(\boldsymbol{x},t) = \frac{1}{\beta_C N!}\int \mathrm{d}\mathscr{C}^N f_C(\boldsymbol{x},t;N)\,\chi_C(\boldsymbol{x};N)\,P(N;t). \tag{2.20}$$

The average thus defined is very useful because it is meaningful also in the case of quantities defined in only one phase, such as the pressure in a liquid–rigid particle suspension. As will be seen shortly, the price to be paid for this attractive feature is the non-commutativity with the operation of differentiation due to the lack of ordinary differentiability of the phase indicator function. In a similar way we introduce the average $\langle f_C\rangle_K(\boldsymbol{x},t\,|\,K)$ conditional on the configuration $\mathscr{C}^K$ of K particles by

$$\langle f_C\rangle_K(\boldsymbol{x},t\,|\,K) = \frac{1}{(N-K)!\,\beta_C^K}\int \mathrm{d}\mathscr{C}^{N-K}\chi_C(\boldsymbol{x};N)f_C(\boldsymbol{x},t;N)\,P(N-K\,|\,K;t). \tag{2.21}$$

It is a simple consequence of these definitions that

$$\langle f_C \rangle (\boldsymbol{x}, t) = \frac{(N-K)!}{\beta_C N!} \int \mathrm{d}\mathscr{C}^K \beta_C^K \langle f_C \rangle_K P(K; t). \tag{2.22}$$

By a calculation similar to that leading to (2.17) one finds

$$\boldsymbol{\nabla}(\beta_C \langle f_C \rangle) = \beta_C \langle \boldsymbol{\nabla} f_C \rangle + \int_{|\boldsymbol{x}-\boldsymbol{y}|=a} \boldsymbol{n}\, \mathrm{d}S_y \int \mathrm{d}^3 w P(1; t) \langle f_C \rangle_1 (\boldsymbol{x}, t \,|\, 1), \tag{2.23}$$

or equivalently, by use of (2.17) to express $\boldsymbol{\nabla}\beta_C$,

$$\boldsymbol{\nabla} \langle f_C \rangle = \langle \boldsymbol{\nabla} f_C \rangle + \frac{1}{\beta_C} \int_{|\boldsymbol{x}-\boldsymbol{y}|=a} \mathrm{d}S_y \boldsymbol{n} \int \mathrm{d}^3 w P(1, t) [\langle f_C \rangle_1 (\boldsymbol{x}, t \,|\, 1) - \langle f_C \rangle (\boldsymbol{x}, t)]. \tag{2.24}$$

To calculate the time derivative we proceed as in (2.18) and find, after an integration by parts in phase space,

$$\frac{\partial}{\partial t}(\beta_C \langle f_C \rangle) = \beta_C \left\langle \frac{\hat{\partial} f_C}{\hat{\partial} t} + \sum_{\alpha=1}^{N} (\boldsymbol{w}^\alpha \cdot \boldsymbol{\nabla}_\alpha f_C + \dot{\boldsymbol{w}}^\alpha \cdot \boldsymbol{\Delta}_\alpha f_C) \right\rangle + \frac{1}{N!} \int \mathrm{d}\mathscr{C}^N f_C P \sum_{\alpha=1}^{N} \boldsymbol{w}^\alpha \cdot \boldsymbol{\nabla}_\alpha \chi_C, \tag{2.25}$$

where the time derivative in the first term has been indicated with a special symbol as a reminder of the fact that it only acts on the explicit time dependence of f_C. This is in fact the partial time derivative of f_C for fixed $\boldsymbol{x}$ and $\mathscr{C}^N$. In the actual flow situation, f_C depends on time not only explicitly, but also through the evolving particle configuration, which is clearly accounted for by the summation over α in the first term in the right-hand side of this equation. Hence we may write

$$\frac{\partial f_C}{\partial t} = \frac{\hat{\partial} f_C}{\hat{\partial} t} + \sum_{\alpha=1}^{N} (\boldsymbol{w}^\alpha \cdot \boldsymbol{\nabla}_\alpha f_C + \dot{\boldsymbol{w}}^\alpha \cdot \boldsymbol{\Delta}_\alpha f_C), \tag{2.26}$$

where now the time derivative in the left-hand side is taken for constant $\boldsymbol{x}$ only. By introducing also the representation (2.16) of $\boldsymbol{\nabla}\chi_C$ we finally have

$$\frac{\partial}{\partial t}(\beta_C \langle f_C \rangle) = \beta_C \left\langle \frac{\partial f_C}{\partial t} \right\rangle - \int \mathrm{d}S_y \int \mathrm{d}^3 w P(1; t) \boldsymbol{n} \cdot \boldsymbol{w} \langle f_C \rangle_1, \tag{2.27}$$

or, using (2.17),

$$\frac{\partial \langle f_C \rangle}{\partial t} = \left\langle \frac{\partial f_C}{\partial t} \right\rangle - \frac{1}{\beta_C} \int \mathrm{d}S_y \int \mathrm{d}^3 w P(1; t) \boldsymbol{n} \cdot \boldsymbol{w} (\langle f_C \rangle_1 - \langle f_C \rangle). \tag{2.28}$$

By use of (2.23) and (2.27) we find

$$\frac{\partial}{\partial t}(\beta_C \langle f_C \rangle) + \boldsymbol{\nabla} \cdot (\beta_C \langle f_C \boldsymbol{u}_C \rangle) = \beta_C \left\langle \frac{\partial f_C}{\partial t} + \boldsymbol{\nabla} \cdot (f_C \boldsymbol{u}_C) \right\rangle - \int_{|\boldsymbol{x}-\boldsymbol{y}|=a} \mathrm{d}S_y \int \mathrm{d}^3 w P(1; t) [\boldsymbol{n} \cdot \boldsymbol{w} \langle f_C \rangle_1 - \boldsymbol{n} \langle f_C \boldsymbol{u}_C \rangle_1]. \tag{2.29}$$

The surface integral is over the surface of all the particles that touch $\boldsymbol{x}$. For all such particles the velocity field satisfies the kinematic boundary condition

$$\boldsymbol{n} \cdot \boldsymbol{w} = \boldsymbol{n} \cdot \boldsymbol{u}_C(\boldsymbol{x}, t \,|\, N), \tag{2.30}$$

and, since $w\langle f_C\rangle_1 = \langle wf_C\rangle_1$, the integral term in (2.29) vanishes identically so that

$$\frac{\partial}{\partial t}(\beta_C\langle f_C\rangle)+\nabla\cdot(\beta_C\langle f_C\boldsymbol{u}_C\rangle)=\beta_C\left\langle\frac{\partial f_C}{\partial t}+\nabla\cdot(f_C\boldsymbol{u}_C)\right\rangle. \tag{2.31}$$

The derivation of the corresponding result for the conditionally averaged f_C is somewhat more involved. Since we only have a limited need for this result in the following, this derivation is presented in Appendix A.

2.2. *Disperse-phase averages*

For a quantity $f_D(\boldsymbol{x},t;N)$ pertaining to the disperse phase the ensemble average is defined analogously to (2.20) as

$$\langle f_D\rangle(\boldsymbol{x},t)=\frac{1}{N!\,\beta_D}\int \mathrm{d}\mathscr{C}^N P(N;t)\,\chi_D f_D(\boldsymbol{x},t;N). \tag{2.32}$$

Conditional averages are defined similarly to (2.21),

$$\langle f_D\rangle_K(\boldsymbol{x},t)=\frac{1}{(N-K)!\,\beta_D^K}\int \mathrm{d}\mathscr{C}^{N-K} P(N-K\,|\,K;t)\,\chi_D f_D(\boldsymbol{x},t;N). \tag{2.33}$$

By using the explicit representation (2.11) of the particle indicator function χ_D, by virtue of the identity of the particles, the average (2.32) may equivalently be written as

$$\langle f_D\rangle(\boldsymbol{x},t)=\frac{1}{(N-1)!\,\beta_D}\int_{|\boldsymbol{y}^{(1)}-\boldsymbol{x}|\leqslant a}\mathrm{d}^3y^{(1)}\int\mathrm{d}^3w^{(1)}\int\mathrm{d}\mathscr{C}^{N-1}P(N;t)\,\chi_D f_D^{(1)}(\boldsymbol{x},t;N), \tag{2.34}$$

where $f_D^{(1)}(\boldsymbol{x},t;N)$ denotes the value of f_D inside the particle centred at $\boldsymbol{y}^{(1)}$ or, with (2.33),

$$\langle f_D\rangle(\boldsymbol{x},t)=\frac{1}{\beta_D}\int_{|\boldsymbol{y}-\boldsymbol{x}|\leqslant a}\mathrm{d}^3y\int\mathrm{d}^3w\beta_D^1\langle f_D^{(1)}\rangle_1 P(1;t). \tag{2.35}$$

If $f_D^{(1)}$ does not depend explicitly on the configuration of the other particles, this relation simplifies to

$$\langle f_D\rangle(\boldsymbol{x},t)=\frac{1}{\beta_D}\int_{|\boldsymbol{y}-\boldsymbol{x}|\leqslant a}\mathrm{d}^3y\int\mathrm{d}^3w\beta_D^1 f_D(\boldsymbol{x},t;1)\,P(1;t). \tag{2.36}$$

It is readily shown from the expression (2.14) for β_D^1 that, owing to the restriction $|\boldsymbol{y}-\boldsymbol{x}|\leqslant a$ on the domain of integration, here $\beta_D^1=1$.

In dealing with the disperse phase, a different kind of average is useful for quantities $g^{(\alpha)}(t;N)$ pertaining to each particle as a whole. Examples may be the centre-of-mass velocity, momentum, orientation, shape parameters (although here we only consider spherical particles), and others. For quantities of this type we define the ensemble average over all the configurations such that one particle centre is at $\boldsymbol{x}$ by

$$\bar{g}(\boldsymbol{x},t)=\frac{1}{n(\boldsymbol{x},t)}\frac{1}{(N-1)!}\int\mathrm{d}^3w^{(1)}\int\mathrm{d}\mathscr{C}^{N-1}P(N;t)\,g^{(1)}(N;t), \tag{2.37}$$

where the identity of the particles has been used. If the quantity $g^{(1)}$ does not depend explicitly on the configuration of the other particles, this definition reduces to

$$\bar{g}(\boldsymbol{x},t)=\frac{1}{n(\boldsymbol{x},t)}\int\mathrm{d}^3wP(\boldsymbol{x},\boldsymbol{w};t)\,g^{(1)}(\boldsymbol{x},\boldsymbol{w},t). \tag{2.38}$$

In the point-particle approximation of (2.12), the average (2.32) becomes identical with (2.37).

From the definition (2.37), after multiplication by n and use of the probability equation (2.2), we have

$$\frac{\partial}{\partial t}(n\bar{g}) = n\overline{\frac{\hat{\partial} g}{\hat{\partial} t}} - \frac{1}{(N-1)!}\int \mathrm{d}^3 w^{(1)} \int \mathrm{d}\mathscr{C}^{N-1} g^{(1)} \sum_{\alpha=1}^{N} [\nabla_\alpha \cdot (w^\alpha P) + \Delta_\alpha \cdot (\dot{w}^\alpha P)]. \quad (2.39)$$

The terms corresponding to $\alpha = 2, 3, \ldots, N$ can be integrated by parts as before and one is left with

$$\frac{\partial}{\partial t}(n\bar{g}) + \nabla \cdot (n\overline{wg}) = n\overline{\dot{g}^{(1)}}, \quad (2.40)$$

where, as in (2.26),

$$\dot{g}^{(1)} = \frac{\hat{\partial} g^{(1)}}{\hat{\partial} t} + \sum_{\alpha=1}^{N} (w^\alpha \cdot \nabla_\alpha g^{(1)} + \dot{w}^\alpha \cdot \Delta_\alpha g^{(1)}). \quad (2.41)$$

Since one may consider the field $g^{(1)}$ as defined at particle centres, this result is the same as a standard one in kinetic theory.

Corresponding transport equations for averages of the type $\langle f_D \rangle$ can also be derived but they will not be needed in the following.

We now apply these relations to the derivation of the equations of motion of the phases.

3. The averaged equations

With the neglect of viscous effects and compressibility, the equations of motion of the continuous phase are

$$\nabla \cdot \boldsymbol{u}_C = 0, \quad (3.1)$$

$$\frac{\partial \boldsymbol{u}_C}{\partial t} + \nabla \cdot (\boldsymbol{u}_C \boldsymbol{u}_C) = -\frac{1}{\rho_C} \nabla p_C + \boldsymbol{g}, \quad (3.2)$$

where $\boldsymbol{u}_C$ and p_C denote the velocity and pressure field, ρ_C is the density, assumed constant, and $\boldsymbol{g}$ a body force. Upon taking $f_C = 1$ in (2.31) and using the continuity equation (3.1) we have

$$\frac{\partial \beta_C}{\partial t} + \nabla \cdot (\beta_C \langle \boldsymbol{u}_C \rangle) = 0. \quad (3.3)$$

By taking $f_C = \boldsymbol{u}_C$, we have from (2.31) and (3.2)

$$\rho_C \frac{\partial}{\partial t}(\beta_C \langle \boldsymbol{u}_C \rangle) + \rho_C \nabla \cdot (\beta_C \langle \boldsymbol{u}_C \rangle \langle \boldsymbol{u}_C \rangle) + \beta_C \nabla \langle p_C \rangle = \beta_C \boldsymbol{A}_C(\boldsymbol{x}, t) + \rho_C \nabla \cdot (\beta_C \boldsymbol{M}_C) + \beta_C \rho_C \boldsymbol{g}, \quad (3.4)$$

where we have introduced the Reynolds-like (kinematic) stress tensor

$$\boldsymbol{M}_C = \langle \boldsymbol{u}_C \rangle \langle \boldsymbol{u}_C \rangle - \langle \boldsymbol{u}_C \boldsymbol{u}_C \rangle = -\langle (\boldsymbol{u}_C - \langle \boldsymbol{u}_C \rangle)(\boldsymbol{u}_C - \langle \boldsymbol{u}_C \rangle) \rangle, \quad (3.5)$$

and we have set

$$\boldsymbol{A}_C(\boldsymbol{x}, t) = \nabla \langle p_C \rangle - \langle \nabla p_C \rangle, \quad (3.6)$$

or, from (2.24),

$$\boldsymbol{A}_C(\boldsymbol{x}, t) = \frac{1}{\beta_C} \int_{|\boldsymbol{x}-\boldsymbol{y}|=a} \boldsymbol{n}\, \mathrm{d}S_y \int P(1; t)\, \mathrm{d}^3 w [\langle p_C \rangle_1 (\boldsymbol{x}, t \mid 1) - \langle p_C \rangle (\boldsymbol{x}, t)]. \quad (3.7)$$

Equation (3.4) has been written in terms of the gradient of the average pressure in conformity with the standard form of the widely used two-fluid models. This step

introduces the quantity A_C defined by (3.6) which must evidently contain inter-phase momentum interactions. Qualitatively, this is so since the first term in A_C is clearly sensitive to the macroscale flow properties, while the second one also contains information on microscale phenomena. As shown in Prosperetti & Jones (1984), this separation of scales is essential for a proper understanding of the role of pressure forces in disperse flows. The formal definition (3.6) has the advantage of introducing a clear operational prescription to effect this scale separation even for concentrated mixtures for which one cannot unambiguously rely on order-of-magnitude arguments. Other approaches to the inter-phase momentum interaction are possible, such as the direct computation of the forces exerted by each phase on the other one (see e.g. Sangani *et al.* 1991 and §7 below). Such calculations can provide useful checks, but it would seem that, ultimately, any two-phase model of engineering significance would have the structure exhibited by (3.4).

If particles collide, a reaction force will also act on the liquid. Formally, this may be considered as arising from the action of forces, highly localized in space and time, appearing in the right-hand side of the Euler equation (3.2). Since we shall have no need for these effects, we have not included them in (3.4). In this connection, see §4 below.

To obtain averaged equations for the disperse phase we use the general relation (2.40). By taking $g^{(1)} = 1$ we have the conservation equation for the particle number density n,

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\bar{\boldsymbol{w}}) = 0. \tag{3.8}$$

Since the particles are assumed to be homogeneous, $\bar{\boldsymbol{w}}$ is the average velocity field of the particles' centres of mass.

In the present inviscid framework, the equation of motion of a particle is

$$m\dot{\boldsymbol{w}} = -\int_{|\boldsymbol{x}-\boldsymbol{z}|=a} \mathrm{d}S_z p_C(\boldsymbol{z}, t; N)\,\boldsymbol{n} + m\boldsymbol{g} + \boldsymbol{F}_c, \tag{3.9}$$

where m is the (constant) particle mass and $\boldsymbol{F}_c$ is the force due to collisions with other particles. By taking in (2.40) $g^{(1)} = m\boldsymbol{w}$ we have

$$\frac{\partial (nm\bar{\boldsymbol{w}})}{\partial t} + \nabla \cdot (nm\,\overline{\boldsymbol{w}\boldsymbol{w}}) = nm\,\overline{\dot{\boldsymbol{w}}}. \tag{3.10}$$

Carrying out the average of (3.9) according to (2.37) and substituting in the right-hand side, we find the averaged momentum equation for the particle with centre of mass at $\boldsymbol{x}$:

$$\frac{\partial (n\bar{\boldsymbol{w}})}{\partial t} + \nabla \cdot (n\,\overline{\boldsymbol{w}\boldsymbol{w}}) = -\frac{1}{m}\int_{|\boldsymbol{x}-\boldsymbol{z}|=a} \mathrm{d}S_z \boldsymbol{n} \int \mathrm{d}^3 w \langle p_c \rangle_1(\boldsymbol{z}, t \mid 1)\, P(\boldsymbol{x}, \boldsymbol{w}; t) + n\boldsymbol{g} + \frac{n}{m}\overline{\boldsymbol{F}_c}. \tag{3.11}$$

With
$$\boldsymbol{M}_D = \bar{\boldsymbol{w}}\,\bar{\boldsymbol{w}} - \overline{\boldsymbol{w}\boldsymbol{w}} = -\overline{(\boldsymbol{w}-\bar{\boldsymbol{w}})(\boldsymbol{w}-\bar{\boldsymbol{w}})}, \tag{3.12}$$

and
$$\boldsymbol{A}_D = \nabla \langle p_C \rangle - \frac{1}{nv}\int_{|\boldsymbol{x}-\boldsymbol{z}|=a} \mathrm{d}S_z \boldsymbol{n} \int \mathrm{d}^3 w P(\boldsymbol{x}, \boldsymbol{w}; t) \langle p_C \rangle_1(\boldsymbol{z}, t \mid 1), \tag{3.13}$$

equation (3.11) becomes, for constant $\rho_D = m/v$,

$$\rho_D \frac{\partial}{\partial t}(n\bar{\boldsymbol{w}}) + \rho_D \nabla \cdot (n\,\bar{\boldsymbol{w}}\,\bar{\boldsymbol{w}}) = -n\nabla \langle p_C \rangle + \rho_D \nabla \cdot (n\boldsymbol{M}_D) + n\rho_D \boldsymbol{g} + n\boldsymbol{A}_D + \frac{n}{v}\overline{\boldsymbol{F}_c}, \tag{3.14}$$

where ρ_D is the density of the disperse-phase material. A subtle but crucial difference between $\boldsymbol{A}_C$ and $\boldsymbol{A}_D$ should be noted here. The first quantity is the result of an integral

over all the particles touching the point $\boldsymbol{x}$. The second one, on the other hand, involves the integral over the surface of a single particle centred at $\boldsymbol{x}$. The two integrals will be equal for a uniform dispersion, but differences are to be expected in the general case. This point will be seen more clearly in the next section where an approximate relation between $\boldsymbol{A}_C$ and $\boldsymbol{A}_D$ is derived. Explicit forms for these two quantities applicable for the dilute case are given in § 5. For future reference we note an alternative form of (3.10) obtained by use of the particle number conservation equation (3.8) and the definition of $\boldsymbol{M}_D$,

$$\overline{\boldsymbol{w}} = \frac{\partial \overline{\boldsymbol{w}}}{\partial t} + \overline{\boldsymbol{w}} \cdot \boldsymbol{\nabla} \overline{\boldsymbol{w}} - \frac{1}{n} \boldsymbol{\nabla} \cdot (n \boldsymbol{M}_D). \tag{3.15}$$

Several analyses in the multiphase flow literature derive the disperse-phase momentum equation by averaging the microscopic equation of motion for the particle material and using the boundary conditions at the surface of each particle. This procedure exactly parallels the treatment given above for the continuous phase and it can easily be developed here as well (Zhang 1993). The approach is, however, much less straightforward and runs into difficulties, e.g. for rigid particles or massless bubbles, and artifices are necessary to deal with these situations (see e.g. Batchelor 1970; Sangani & Didwania 1993*a*). In contrast, we have chosen to treat the particles as entities in themselves at the outset and to obtain the average disperse-phase momentum equation by averaging their equation of motion directly. This approach is evidently more 'economical' in the same sense as in ordinary mechanics where the description of a rigid-body motion in terms of its independent degrees of freedom is simpler than in terms of the motion of its individual material points because the infinitely many kinematic constraints of rigidity are automatically accounted for. Clearly, this method can be extended to any situation in which there are 'attributes' that can be ascribed to the individual particles. For example, for spherical expanding or contracting bubbles, one can attribute a value of the radius to each bubble and average the corresponding evolution equation as shown in Zhang & Prosperetti (1994). More general situations could be handled by ascribing to each bubble shape parameters and averaging the corresponding evolution equations. As another example, for rigid particles with complex shapes, in addition to the centre-of-mass momentum equation, one would average the single-particle angular momentum equation. In this way, the disperse phase would be described by the field of average centre velocities $\overline{\boldsymbol{w}}$ and the field of average angular velocities. Such an approach for the case of ellipsoidal particles is currently being developed.

The set of averaged equations derived above is not closed since the right-hand sides contain averages with one particle held fixed and averages of products that must be expressed as products of averages. Both these difficulties are, of course, well known. In principle, one can generate averaged equations correct to any order $(\beta_D)^K$ of the disperse-phase volume fraction by writing down the equations averaged with K particles held fixed, and realizing that the terms involving $K+1$ fixed particles that appear give rise to contributions of higher order and can therefore be neglected. With this approximation the order-K equations are closed and can be solved. These solutions are then used to calculate the averages with K particles fixed that appear in the equations of order $K-1$, and so on. In this way a closed set of equations for the unconditionally averaged fields can in principle be obtained. We demonstrate the first step of this closure procedure in § 5. While it is doubtful that such a plan can be carried out beyond, say, $O(\beta_D)^2$, our equations also provide a useful basis for the analysis of the results of numerical simulations as will be explained in § 10.

Since we are neglecting the viscosity of the continuous phase, it is interesting to consider whether a microscopically irrotational flow will retain this property after averaging. By use of the differentiation rule (2.24) one finds

$$\nabla \times \langle \boldsymbol{u}_C \rangle = \langle \nabla \times \boldsymbol{u}_C \rangle - \frac{1}{\beta_C} \int \mathrm{d}^3 w \int \mathrm{d}S_y \, P(1;t) (\langle \boldsymbol{u}_C \rangle_1 - \langle \boldsymbol{u}_C \rangle) \times \boldsymbol{n}. \tag{3.16}$$

This relation shows that irrotationality is maintained to $O(1)$, but not to $O(\beta_D)$. The delicate local balance responsible for giving $\nabla \times \boldsymbol{u}_C = 0$ is upset by the averaging process. This result may be interpreted physically as follows. A component of the vorticity at $\boldsymbol{x}$ is essentially proportional to the circulation of the velocity field around a small contour centred at $\boldsymbol{x}$. Since the flow is irrotational, this circulation is zero for all the realizations where the contour lies wholly in the continuous phase. However, for some realizations, the contour will be interrupted by the presence of particles, and in this case the integration will not give a vanishing result for a non-uniform state.

A similar calculation gives, using the fact that $\boldsymbol{w}$ is an independent variable,

$$\nabla \times \bar{\boldsymbol{w}} = -\frac{1}{n} \int (\boldsymbol{w} - \bar{\boldsymbol{w}}) \times \nabla P(\boldsymbol{x}, \boldsymbol{w}; t) \, \mathrm{d}^3 w. \tag{3.17}$$

This result would also hold in the presence of viscous flow inside the particles as $\boldsymbol{w}$ is not the velocity field in the particle phase, but the velocity of the centre of mass of the particles. Contrary to the continuous-phase average vorticity relation (3.16), the rotational character of the mean centre-of-mass particle velocity field depends on the probability distribution and does not necessarily vanish to $O(1)$.

4. Small-particle approximation

The equations derived in the preceding sections contain several terms involving integration over spheres with a radius equal to the particle radius a. When the averaged quantities vary slowly on this scale, these integrals may be approximated by Taylor series expansions.

For the disperse-phase volume fraction (2.13) this procedure leads to

$$\beta_D(\boldsymbol{x}, t) = v[n(\boldsymbol{x}, t) + \tfrac{1}{10} a^2 \nabla^2 n(\boldsymbol{x}, t) + O(a^4)]. \tag{4.1}$$

This relation gives the leading-order correction term to the widely used approximation $\beta_D = nv$.

In a similar fashion we can obtain an approximate relation between the ensemble-average disperse-phase velocity $\langle \boldsymbol{u}_D \rangle (\boldsymbol{x}, t)$ defined according to (2.32) or (2.35) and the mean particle centre-of-mass velocity $\bar{\boldsymbol{w}}(\boldsymbol{x}, t)$. Since here we confine ourselves to rigid particles, the velocity field $\boldsymbol{u}_D^{(\alpha)}(\boldsymbol{x}, t; N)$ inside the αth particle is given by

$$\boldsymbol{u}_D^{(\alpha)}(\boldsymbol{x}, t; N) = \boldsymbol{w}^{(\alpha)} + \boldsymbol{\Omega}^{(\alpha)} \times (\boldsymbol{x} - \boldsymbol{y}^{(\alpha)}), \tag{4.2}$$

where $\boldsymbol{\Omega}^{(\alpha)}$ is the angular velocity around the instantaneous axis of rotation through the particle centre $\boldsymbol{y}^{\alpha}$. Upon substitution into (2.36) one finds

$$\langle \boldsymbol{u}_D \rangle (\boldsymbol{x}, t) = \frac{1}{\beta_D} \int_{|\boldsymbol{x}-\boldsymbol{y}| \leqslant a} n(\boldsymbol{y}, t) \, [\bar{\boldsymbol{w}}(\boldsymbol{y}, t) + \bar{\boldsymbol{\Omega}}(\boldsymbol{y}, t) \times (\boldsymbol{x} - \boldsymbol{y})] \, \mathrm{d}^3 y. \tag{4.3}$$

At this point a Taylor series expansion centred at $\boldsymbol{x}$ gives

$$\langle \boldsymbol{u}_D \rangle (\boldsymbol{x}, t) = \bar{\boldsymbol{w}}(\boldsymbol{x}, t) + \tfrac{1}{10} a^2 \left[\nabla^2 \bar{\boldsymbol{w}} + \nabla \times \bar{\boldsymbol{\Omega}} + \frac{1}{n} (\nabla n \cdot \nabla \bar{\boldsymbol{w}} + \nabla n \times \bar{\boldsymbol{\Omega}}) + O(a^2) \right]. \tag{4.4}$$

The evaluation of the correction term in square brackets requires a dynamical equation for $\boldsymbol{\Omega}$. For the present inviscid flow situation, if the particles are smooth so that no tangential force arises upon collision, clearly $\dot{\boldsymbol{\Omega}}^{(\alpha)} = 0$. This may be used as a starting point for the derivation. Since we shall have no use for this result, we do not pursue the matter further. It may be noted, however, that the dependence of $\langle \boldsymbol{u}_D \rangle$ on $\boldsymbol{\Omega}^{(\alpha)}$, which, in the present hypotheses, is an irrelevant quantity as far as the average linear momentum of the disperse phase is concerned, lends further support to the point of view adopted in the present paper according to which $\bar{\boldsymbol{w}}$, rather than $\langle \boldsymbol{u}_D \rangle$, is the fundamental disperse-phase velocity field. Indeed, it is evident that, by a suitable manipulation of the initial conditions for $\boldsymbol{\Omega}^{(\alpha)}$, $\langle \boldsymbol{u}_D \rangle$ can be made to differ arbitrarily much from $\bar{\boldsymbol{w}}$, even though this difference can in no way alter the motion of the particles. If the particle momentum equation were based on $\langle \boldsymbol{u}_D \rangle$, one would have to include the terms necessary to cancel this spurious effect.

Expanding the integrand we also readily find

$$\frac{1}{v}\int_{|\boldsymbol{x}-\boldsymbol{z}|=a} \mathrm{d}S_z \boldsymbol{n} \langle p_C \rangle (\boldsymbol{z}, t) = \boldsymbol{\nabla} \langle p_C \rangle + \frac{\pi}{10} a^2 \boldsymbol{\nabla} [\nabla^2 \langle p_C \rangle (\boldsymbol{x}, t)], \tag{4.5}$$

with an error of order a^4. With this relation, and dropping terms of order a^2 and higher, the expression (3.13) for $\boldsymbol{A}_D$ may be written as

$$\beta_D \boldsymbol{A}_D(\boldsymbol{x}, t) = -\int \mathrm{d}^3 w P(\boldsymbol{x}, \boldsymbol{w}; t) \int_{|\boldsymbol{x}-\boldsymbol{z}|=a} \mathrm{d}S_z \boldsymbol{n} [\langle p_C \rangle_1 (\boldsymbol{z}, t \,|\, \boldsymbol{x}, \boldsymbol{w}) - \langle p_C \rangle (\boldsymbol{z}, t)], \tag{4.6}$$

or, upon setting $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{s}$,

$$\beta_D \boldsymbol{A}_D(\boldsymbol{x}, t) = -\int_{|\boldsymbol{s}|=a} \mathrm{d}S_s \boldsymbol{n} \int \mathrm{d}^3 w P(\boldsymbol{x}, \boldsymbol{w}; t) [\langle p_C \rangle_1 (\boldsymbol{x}+\boldsymbol{s}, t \,|\, \boldsymbol{x}, \boldsymbol{w}) - \langle p_C \rangle (\boldsymbol{x}+\boldsymbol{s}, t)]. \tag{4.7}$$

Using this expression as a starting point we may derive an approximate relation between $\boldsymbol{A}_D$ and $\boldsymbol{A}_C$ and simplify the expression of the latter. To explain the procedure (Hinch 1977; Sangani & Didwania 1993*a*), consider the term $\langle p_C \rangle_1 (\boldsymbol{x}, t \,|\, 1) \equiv \langle p_C \rangle_1 (\boldsymbol{x}, t \,|\, \boldsymbol{y}, \boldsymbol{w})$ of the integrand in the expression (3.7) of $\boldsymbol{A}_C$. Since in the integral $|\boldsymbol{x}-\boldsymbol{y}| = a$, this quantity may be also written as $\langle p_C \rangle_1 (\boldsymbol{y}+\boldsymbol{s}, t \,|\, \boldsymbol{y}, \boldsymbol{w})$, with $\boldsymbol{y} = \boldsymbol{x}+\boldsymbol{s}$ and $|\boldsymbol{s}| = a$. In the absence of extreme gradients of the average fields, this quantity depends slowly on $\boldsymbol{y}$, which denotes the particle centre, and rapidly on $\boldsymbol{s}$, which is the position relative to the centre. This consideration justifies a series expansion in $\boldsymbol{y}$ around $\boldsymbol{x}$ for fixed $\boldsymbol{s}$:

$$\begin{aligned} P(1;t)[\langle p_C \rangle_1 (\boldsymbol{x}, t \,|\, 1) - \langle p_C \rangle (\boldsymbol{x}, t)] \\ &= P(\boldsymbol{y}, \boldsymbol{w}; t)[\langle p_C \rangle_1 (\boldsymbol{y}+\boldsymbol{s}, t \,|\, \boldsymbol{y}, \boldsymbol{w}) - \langle p_C \rangle (\boldsymbol{y}+\boldsymbol{s}, t)] \\ &\approx P(\boldsymbol{x}, \boldsymbol{w}; t)[\langle p_C \rangle_1 (\boldsymbol{x}+\boldsymbol{s}, t \,|\, \boldsymbol{x}, \boldsymbol{w}) - \langle p_C \rangle (\boldsymbol{x}+\boldsymbol{s}, t)] \\ &\quad + \boldsymbol{s} \cdot \boldsymbol{\nabla}_x \{P(\boldsymbol{x}, \boldsymbol{w}; t)[\langle p_C \rangle_1 (\boldsymbol{x}+\boldsymbol{s}, t \,|\, \boldsymbol{y}, \boldsymbol{w}) - \langle p_C \rangle (\boldsymbol{x}+\boldsymbol{s}, t)]\}. \end{aligned} \tag{4.8}$$

Upon substitution into (3.7) and comparison with (4.7), we then find

$$\beta_C \boldsymbol{A}_C = -\beta_D \boldsymbol{A}_D + \boldsymbol{\nabla} \cdot (\beta_D \boldsymbol{T}_C), \tag{4.9}$$

where the additional stress tensor $\boldsymbol{T}_C$ is defined by

$$\boldsymbol{T}_C = -\frac{a}{\beta_D} \int \mathrm{d}^3 w P(\boldsymbol{x}, \boldsymbol{w}, t) \int_{|\boldsymbol{x}-\boldsymbol{z}|=a} \mathrm{d}S_z \boldsymbol{n}\boldsymbol{n} [\langle p_C \rangle_1 (\boldsymbol{z}, t \,|\, \boldsymbol{x}, \boldsymbol{w}) - \langle p_C \rangle (\boldsymbol{x}, t)]. \tag{4.10}$$

For a general case in which truncation of the expansion to the second term as in (4.8) is invalid, one might regard (4.9) as the *definition* of $\boldsymbol{T}_C$. That the sum $\beta_C \boldsymbol{A}_C + \beta_D \boldsymbol{A}_D$ can be expressed as a divergence would then follow from Lagrange's form of Taylor's theorem, the applicability of which only rests on the existence of the first derivatives of $\langle p_C \rangle_1$.

For purposes of comparison with other investigators' results it is useful to exhibit explicitly the form of the momentum equations obtained by adopting the small-particle approximations developed here. They are

$$\rho_D \frac{\partial}{\partial t}(\beta_D \bar{\boldsymbol{w}}) + \rho_D \nabla \cdot (\beta_D \bar{\boldsymbol{w}} \bar{\boldsymbol{w}}) + \beta_D \nabla \langle p_C \rangle = \beta_D \boldsymbol{A}_D(\boldsymbol{x}, t) + \rho_D \nabla \cdot (\beta_D \boldsymbol{M}_D) + \beta_D \rho_D \boldsymbol{g} + n \bar{\boldsymbol{F}}_c, \quad (4.11)$$

$$\rho_C \frac{\partial}{\partial t}(\beta_C \langle \boldsymbol{u}_C \rangle) + \rho_C \nabla \cdot (\beta_C \langle \boldsymbol{u}_C \rangle \langle \boldsymbol{u}_C \rangle) + \beta_C \nabla \langle p_C \rangle = -\beta_D \boldsymbol{A}_D(\boldsymbol{x}, t) + \nabla \cdot (\beta_C \rho_C \boldsymbol{M}_C + \beta_D \boldsymbol{T}_C) + \beta_C \rho_C \boldsymbol{g} - \langle \boldsymbol{F}_{cL} \rangle, \quad (4.12)$$

where $-\langle \boldsymbol{F}_{cL} \rangle$ is the mean force on the liquid due to particle collisions. From the last equation one can gain a further understanding of the physical origin of the stress $\boldsymbol{T}_C$. Consider a surface element $\mathrm{d}S$ in the mixture around a point $\boldsymbol{x}$ and the force that the mixture on one side of $\mathrm{d}S$ exerts on the mixture on the other side. This force differs from that due to the average pressure $\langle p_C \rangle$ because $\boldsymbol{x}$ lies on a particle surface with a finite probability. Suppose that the particle velocity distribution is biased toward a certain direction. If $\mathrm{d}S$ is taken parallel to this direction, in view of the pressure distribution around a particle in potential flow, the extra pressure on it will tend to be lower than $\langle p_C \rangle$. On the other hand, if $\mathrm{d}S$ is taken orthogonal to the direction of motion, $\boldsymbol{x}$ tends to be close to the stagnation point on the particle surface and therefore the contribution to the pressure force will be higher than $\langle p_C \rangle$. The dependence of the force on the orientation of $\mathrm{d}S$ is characteristic of the action of a stress.

Although one might expect the two collision contributions to balance in the uniform case, we avoid enforcing this condition in general as the same mechanism that makes $\beta_C \boldsymbol{A}_C$ different from $\beta_D \boldsymbol{A}_D$ might be at work here.

5. Averaged equations to first order

We study here the dilute limit in which the particles introduce an $O(\beta_D)$ correction to the continuous-phase equations. This case is interesting not only because it is the simplest one in which the closure program outlined above can rigorously be carried out, but also because it gives some insight into the structure of the equations that may prove useful in future numerical investigations of the finite-volume-fraction case.

According to the definitions (3.7) and (3.13), the evaluation of $\boldsymbol{A}_C$ and $\boldsymbol{A}_D$ requires the conditionally averaged pressure field $\langle p_C \rangle_1$. Since in the continuous-phase momentum equation $\langle p_C \rangle_1$ appears in a term of $O(\beta_D)$ (cf. (3.7)), for results accurate to $O(\beta_D)$ it is sufficient to calculate this quantity correct to $O(1)$. The pertinent equations are given in Appendix A and, to $O(1)$ and with the neglect of the body force, they are

$$\nabla \cdot \langle \boldsymbol{u}_C \rangle_1 = 0, \quad (5.1)$$

$$\rho_C \left[\frac{\partial \langle \boldsymbol{u}_C \rangle_1}{\partial t} + \langle \boldsymbol{u}_C \rangle_1 \cdot \nabla \langle \boldsymbol{u}_C \rangle_1 \right] + \nabla \langle p_C \rangle_1 = 0. \quad (5.2)$$

All the fields are averaged conditionally to the presence of a particle centred at $\boldsymbol{y}$ with velocity $\boldsymbol{w}$. For $|\boldsymbol{x}-\boldsymbol{y}|\to\infty$, the effect of this particle becomes weaker so that (Batchelor 1972)

$$\langle \boldsymbol{u}_C\rangle_1 \to \langle \boldsymbol{u}_C\rangle, \quad \langle p_C\rangle_1 \to \langle p_C\rangle. \tag{5.3}$$

Upon taking the conditional average of (2.30) we find the following form of the kinematic boundary condition to be applied at the surface of the particle:

$$\langle \boldsymbol{u}_C\rangle_1 \cdot \boldsymbol{n} = \boldsymbol{w}\cdot\boldsymbol{n}. \tag{5.4}$$

The solution of this problem is given in detail in Appendix B. It is based on the representation of $\langle \boldsymbol{u}_C\rangle_1$ in terms of a scalar potential since departures from irrotationality are of higher order in β_D as shown by (3.16). For the disperse-phase term $\boldsymbol{A}_D$ we find

$$\boldsymbol{A}_D = \tfrac{1}{2}\rho_C\left[\frac{\partial\langle \boldsymbol{u}_C\rangle}{\partial t} + \langle \boldsymbol{u}_C\rangle\cdot\nabla\langle \boldsymbol{u}_C\rangle - \frac{\partial}{\partial t}\langle \boldsymbol{u}_D\rangle - \langle \boldsymbol{u}_D\rangle\cdot\nabla\langle \boldsymbol{u}_D\rangle + \frac{1}{\beta_D}\nabla(\beta_D \boldsymbol{M}_D)\right]. \tag{5.5}$$

Aside from the Reynolds-stress-like term $\boldsymbol{M}_D$, this expression agrees with the result for the force on a single sphere immersed in a flow obtained by several researchers and, most recently, by Auton, Hunt & Prud'homme (1988). Here and in the rest of this section we write $\langle \boldsymbol{u}_D\rangle$ in place of $\bar{\boldsymbol{w}}$ which is permissible to $O(\beta_D)$ on the basis of (4.4).

In Appendix B we find the following expression for the stress tensor $\boldsymbol{T}_C$ appearing in the expression (4.9) of $\boldsymbol{A}_C$:

$$\begin{aligned}\boldsymbol{T}_C = {}& \tfrac{1}{5}\rho_C[2(\langle \boldsymbol{u}_C\rangle - \langle \boldsymbol{u}_D\rangle)^2 \boldsymbol{I} - \tfrac{9}{4}(\langle \boldsymbol{u}_C\rangle - \langle \boldsymbol{u}_D\rangle)(\langle \boldsymbol{u}_C\rangle - \langle \boldsymbol{u}_D\rangle)] \\ & - \tfrac{1}{5}\rho_C[2(\mathrm{Tr}\,\boldsymbol{M}_D)\boldsymbol{I} - \tfrac{9}{4}\boldsymbol{M}_D].\end{aligned} \tag{5.6}$$

This result is valid to $O(1)$ and is obtained by expanding $\langle \boldsymbol{u}_C\rangle$ in Taylor series about a particle centre. Accordingly the error is of the order of the square of the ratio of the particle size to the macroscopic lengthscale. As a check of the approximation used in (4.8), we have calculated $\boldsymbol{A}_C$ directly from its definition (3.6), correct to $O(\beta_D)$, finding a result identical to (4.9) with $\boldsymbol{T}_C$ given by (5.6).

In Appendix B we also find, to the same degree of approximation,

$$\begin{aligned}\boldsymbol{M}_C = {}& -\tfrac{1}{20}\beta_D[3(\langle \boldsymbol{u}_C\rangle - \langle \boldsymbol{u}_D\rangle)\cdot(\langle \boldsymbol{u}_C\rangle - \langle \boldsymbol{u}_D\rangle)\boldsymbol{I} \\ & + (\langle \boldsymbol{u}_C\rangle - \langle \boldsymbol{u}_D\rangle)(\langle \boldsymbol{u}_C\rangle - \langle \boldsymbol{u}_D\rangle)] + \tfrac{1}{20}\beta_D[3(\mathrm{Tr}\,\boldsymbol{M}_D)\boldsymbol{I} + \boldsymbol{M}_D].\end{aligned} \tag{5.7}$$

The terms in the first square brackets are the same as the result obtained by Biesheuvel & van Wijngaarden (1984), who, however, implicitly assumed the bubbles to move with the same velocity, which implies $\boldsymbol{M}_D = 0$.

While the previous results for $\boldsymbol{M}_C$ and $\boldsymbol{T}_C$ are frame-indifferent, the expression (5.5) for $\boldsymbol{A}_D$ is not. This fact is not surprising as, in the derivation, the flow is assumed to be irrotational, a fact that forbids the use of a rotating coordinate system. The previous expression can, however, easily be corrected so as to be applicable to a frame rotating with angular velocity $\boldsymbol{\Omega}$. Indeed, since both the gradient operator and p_C are frame-indifferent, it follows from the definitions (3.6), (3.13) that so must be $\boldsymbol{A}_C$ and $\boldsymbol{A}_D$. Hence we add to each one of these quantities a new term to be determined in such a way that, upon changing from an inertial to a rotating frame, the result is frame-indifferent. The calculation is given in Appendix B. The result is unique and is found to be

$$\boldsymbol{A}'_D = \boldsymbol{A}_D - \rho_C\,\boldsymbol{\Omega}\times(\langle \boldsymbol{u}_D\rangle - \langle \boldsymbol{u}_C\rangle), \tag{5.8}$$

where the first term in the right-hand side is given by (5.5).

If we assume objectivity (Drew, Cheng & Lahey 1979), this result is also applicable to the case of a mixture rotating with the angular velocity $-\boldsymbol{\Omega}$, which may be approximated by $-\frac{1}{2}\nabla\times\langle\boldsymbol{u}_C\rangle$ provided this quantity is slowly varying in space. We thus have

$$\boldsymbol{A}'_D = \boldsymbol{A}_D + \tfrac{1}{2}\rho_C(\nabla\times\langle\boldsymbol{u}_C\rangle)\times(\langle\boldsymbol{u}_D\rangle-\langle\boldsymbol{u}_C\rangle). \tag{5.9}$$

The corresponding expression for $\boldsymbol{A}'_C$ follows from (4.9) with $\boldsymbol{A}'_D$ in place of $\boldsymbol{A}_D$. The additional term in (5.9) is the familiar lift force (see e.g. Auton *et al.* 1988). The recovery of this result confirms the validity of the objectivity principle in this case.

Combining the previous results we have the final form of the momentum equations. For the disperse phase we find

$$\begin{aligned}
&\rho_D\beta_D\left[\frac{\partial\langle\boldsymbol{u}_D\rangle}{\partial t}+(\langle\boldsymbol{u}_D\rangle\cdot\nabla)\langle\boldsymbol{u}_D\rangle\right]+\beta_D\nabla\langle p_C\rangle\\
&\quad=\tfrac{1}{2}\rho_C\beta_D\left[\frac{\partial\langle\boldsymbol{u}_C\rangle}{\partial t}+\langle\boldsymbol{u}_C\rangle\cdot\nabla\langle\boldsymbol{u}_C\rangle-\frac{\partial}{\partial t}\langle\boldsymbol{u}_D\rangle-\langle\boldsymbol{u}_D\rangle\cdot\nabla\langle\boldsymbol{u}_D\rangle\right]\\
&\qquad+\tfrac{1}{2}\rho_C\beta_D(\nabla\times\langle\boldsymbol{u}_C\rangle)\times(\langle\boldsymbol{u}_D\rangle-\langle\boldsymbol{u}_C\rangle)+(\rho_D+\tfrac{1}{2}\rho_C)\nabla\cdot(\beta_D\boldsymbol{M}_D)+\beta_D\rho_D\boldsymbol{g}.
\end{aligned}\tag{5.10}$$

Similarly, the final form of the continuous-phase momentum equation is

$$\begin{aligned}
&\rho_C\beta_C\left[\frac{\partial\langle\boldsymbol{u}_C\rangle}{\partial t}+(\langle\boldsymbol{u}_C\rangle\cdot\nabla)\langle\boldsymbol{u}_C\rangle\right]+\beta_C\nabla\langle\rho_C\rangle\\
&\quad=-\tfrac{1}{2}\rho_C\beta_D\left[\frac{\partial\langle\boldsymbol{u}_C\rangle}{\partial t}+\langle\boldsymbol{u}_C\rangle\cdot\nabla\langle\boldsymbol{u}_C\rangle-\frac{\partial\langle\boldsymbol{u}_D\rangle}{\partial t}-\langle\boldsymbol{u}_D\rangle\cdot\nabla\langle\boldsymbol{u}_D\rangle\right]\\
&\qquad-\tfrac{1}{2}\rho_C\beta_D(\nabla\times\langle\boldsymbol{u}_C\rangle)\times(\langle\boldsymbol{u}_D\rangle-\langle\boldsymbol{u}_C\rangle)+\tfrac{1}{4}\rho_C\nabla\cdot\{\beta_D[(\langle\boldsymbol{u}_C\rangle-\langle\boldsymbol{u}_D\rangle)^2\boldsymbol{I}\\
&\qquad-2(\langle\boldsymbol{u}_C\rangle-\langle\boldsymbol{u}_D\rangle)(\langle\boldsymbol{u}_C\rangle-\langle\boldsymbol{u}_D\rangle)]\}-\tfrac{1}{4}\rho_C\nabla[\beta_D(\operatorname{Tr}\boldsymbol{M}_D)]+\beta_C\rho_C\boldsymbol{g}.
\end{aligned}\tag{5.11}$$

A comparison with the results of others and a discussion of these equations is given in §10.

Closure of the system requires an expression for the fluctuating particle volume flux tensor $\boldsymbol{M}_D$ (see e.g. Drew 1991). This missing information cannot be supplied internally by the theory without a specification of the initial conditions imposed on the particle probability distribution. This point was noted by Biesheuvel & Spoelstra (1989) who explicitly assumed that, at each position and time, the particle velocity probability distribution is strongly peaked around its local, instantaneous mean value. In this case $\boldsymbol{M}_D = 0$. A similar assumption – whether explicit or implicit – seems to be present in most of the previous work.

6. The linear problem at finite volume fractions

We now consider the case of finite volume fractions for the linear problem for which the momentum equations simplify to

$$\rho_C\frac{\partial}{\partial t}\langle\boldsymbol{u}_C\rangle+\nabla\langle p_C\rangle=\boldsymbol{A}_C(\boldsymbol{x},t)+\rho_C\boldsymbol{g}, \tag{6.1}$$

$$\rho_D\frac{\partial}{\partial t}\bar{\boldsymbol{w}}+\nabla\langle p_C\rangle=\boldsymbol{A}_D(\boldsymbol{x},t)+\rho_D\boldsymbol{g}. \tag{6.2}$$

With the assumption of weak non-homogeneity, i.e. that the volume fraction is nearly uniform, one can determine on the basis of rather general considerations the form of 'constitutive laws' for A_C and A_D.

Let us consider A_D first. The primary variables in our system are the average pressure $\langle p_C \rangle$, the average velocities $\langle \boldsymbol{u}_C \rangle$ and $\bar{\boldsymbol{w}}$, and the volume fraction of the disperse phase β_D. If the momentum equations (6.1), (6.2), together with the continuity equations, are to form a closed system, then A_D must be expressible in terms of these quantities and, possibly, their derivatives. We can however disregard $\langle p_C \rangle$ because, for an incompressible fluid, pressure is only defined up to an arbitrary constant that the explicit form (3.13) for A_D proves to be irrelevant, and $\nabla \langle p_C \rangle$ can also be disregarded because it can be expressed in terms of the other variables from the momentum equations. Similarly, we can neglect the body force $\boldsymbol{g}$ because it can be absorbed in p_C by a suitable redefinition of this quantity that would leave A_D unchanged, as readily follows from its definition.

The fact that we are dealing with a linear problem without dissipation rules out the presence of the velocities $\langle \boldsymbol{u}_C \rangle$, $\bar{\boldsymbol{w}}$ and of their spatial derivatives because they do not have the correct behaviour under time reversal. A term of the form $\langle \boldsymbol{u}_C \rangle \partial \beta_D / \partial t$ has the correct time-reversal symmetry but, upon use of the continuity equations, it is seen to be quadratic in the velocities and therefore negligible under the present approximations.

One is then left with a general structure that may be written in the form

$$A_D = \tfrac{1}{2} \rho_C \beta_C \left[\boldsymbol{C} \cdot \frac{\partial}{\partial t} (\langle \boldsymbol{u}_C \rangle - \bar{\boldsymbol{w}}) + \boldsymbol{E} \cdot \frac{\partial}{\partial t} (\langle \boldsymbol{u}_C \rangle + \bar{\boldsymbol{w}}) \right], \tag{6.3}$$

where the coefficients $\boldsymbol{C}, \boldsymbol{E}$ are second-order tensors that can depend upon $\rho_D/\rho_C, \beta_D$, $\nabla \beta_D$. In this list we do not include $\partial \beta_{C,D} / \partial t$ because of linearity and time-reversal invariance. In the assumption of weak spatial gradients of the volume fractions we allow terms that are at most linear in $\nabla \beta_D = -\nabla \beta_C$. However, it is impossible to linearly build a second-order tensor with the vector $\nabla \beta_D$, so that $\boldsymbol{C}$ and $\boldsymbol{E}$ must be isotropic, e.g. $\boldsymbol{C} = C\boldsymbol{I}$. Furthermore, if the frame of reference is changed to one in arbitrary rectilinear motion with respect to the original one, the local value of p_C cannot be affected, and therefore, from its definition, neither can A_D. Alternatively, an apparent force of constant direction can be absorbed in the body force $\boldsymbol{g}$ and therefore cannot affect A_D. Hence $\boldsymbol{E}$ must vanish and one is left with

$$A_D = \tfrac{1}{2} \rho_C \beta_C C(\beta_D, \rho_D/\rho_C) \frac{\partial}{\partial t} (\langle \boldsymbol{u}_C \rangle - \bar{\boldsymbol{w}}). \tag{6.4}$$

Here the leading terms that have been neglected are proportional to $|\nabla \beta_{C,D}|^2$ and $\nabla^2 \beta_{C,D}$, and hence are proportional to the inverse square of the macroscopic length L associated with the particle distribution.

A similar argument applied to $\boldsymbol{T}_C$ gives

$$T_{Cij} = \rho_C a^2 \left\{ D_1 \left[\nabla \beta_D \cdot \frac{\partial}{\partial t} (\langle \boldsymbol{u}_C \rangle - \bar{\boldsymbol{w}}) \right] \delta_{ij} + \tfrac{1}{2} D_2 \left[\frac{\partial \beta_D}{\partial x_i} \frac{\partial}{\partial t} (\langle u_{Cj} \rangle - \bar{w}_j) + \frac{\partial \beta_D}{\partial x_j} \frac{\partial}{\partial t} (\langle u_{Ci} \rangle - w_i) \right] \right\}, \tag{6.5}$$

where $D_{1,2} = D_{1,2}(\beta_D, \rho_D/\rho_C)$. The dilute-limit results of the previous section show that both D_1 and D_2 vanish for $\beta_D = 0$. With A_D given by (6.4), equation (4.9) provides an expression for A_C. Actually, the divergence of $\boldsymbol{T}_C$ rather than $\boldsymbol{T}_C$ itself enters in this

relation. Therefore, whenever the characteristic length associated with this divergence is comparable with, or longer than, that associated with the β_D distribution, $\nabla\cdot(\beta_D \boldsymbol{T}_C)$ is of the same order as the terms that have been neglected in the previous developments and can be disregarded. In this case, to a consistent approximation,

$$\boldsymbol{A}_C = \tfrac{1}{2}\rho_C \beta_D C(\beta_D, \rho_D/\rho_C)\frac{\partial}{\partial t}(\bar{\boldsymbol{w}} - \langle \boldsymbol{u}_C\rangle). \tag{6.6}$$

We shall assume this form in the following. (With this hypothesis about characteristic lengthscales, as shown by (4.4), to the same accuracy we can also assume $\bar{\boldsymbol{w}} = \langle \boldsymbol{u}_D\rangle$. Since the angular momentum of the particles is conserved, there is no loss of generality in assuming a vanishing initial particle angular velocity for the present purposes.)

The coefficient C appearing in (6.4), (6.6) is related to the added-mass coefficient variously defined in the literature, as we show in the next section. Under the hypotheses outlined above, for results correct to $O(\beta_D)$ included, it is sufficient to determine C for the case of a uniform mixture, which will be accomplished by direct numerical simulation in the following sections.

7. Connection with the added-mass coefficient

The previous results are related to earlier work devoted to the added-mass interaction in an inviscid incompressible liquid containing a collection of spheres as we now describe.

In a recent paper, Sangani *et al.* (1991) considered the linear oscillatory motion of a suspension of rigid spheres at large Reynolds numbers, a problem that is equivalent, as is easily shown (Sangani & Prosperetti 1993), to the case of impulsive motion studied by van Wijngaarden (1976). Motivated by the expression for the force $\boldsymbol{F}$ acting on a single particle with velocity $\boldsymbol{w}$ immersed in an incompressible inviscid flow with velocity $\boldsymbol{u}_\infty$ at infinity,

$$\boldsymbol{F} = \tfrac{1}{2}\rho_C v(\dot{\boldsymbol{u}}_\infty - \dot{\boldsymbol{w}}) + \rho_C v\dot{\boldsymbol{u}}_\infty, \tag{7.1}$$

Sangani *et al.* express the average force per particle in the form

$$\langle \boldsymbol{F}\rangle = \tfrac{1}{2}C_a \rho_C v\frac{\partial}{\partial t}(\langle \boldsymbol{u}_m\rangle - \bar{\boldsymbol{w}}) + \rho_C v\frac{\partial}{\partial t}\langle \boldsymbol{u}_m\rangle, \tag{7.2}$$

where C_a is defined as the added-mass coefficient and $\langle \boldsymbol{u}_m\rangle$ is the average volumetric flow rate of the mixture

$$\langle \boldsymbol{u}_m\rangle = \beta_C\langle \boldsymbol{u}_C\rangle + \beta_D\bar{\boldsymbol{w}}. \tag{7.3}$$

It should be noted that, by the manner of its derivation (see e.g. Landau & Lifshitz 1959), (7.1) presupposes a locally uniform pressure gradient acting on the liquid. Similarly, it is only under these hypotheses that (7.2) can be considered a correct representation of the force on the disperse phase and, indeed, Sangani *et al.* (1991) limit their considerations to the case $\boldsymbol{g} = 0$ with a microscopic pressure field of the special form

$$p_C(\boldsymbol{x}, t; N) = -\boldsymbol{G}(t)\cdot\boldsymbol{x} + p_C^N(\boldsymbol{x} - \mathscr{C}^N; t), \tag{7.4}$$

where, in the last term, $\boldsymbol{x} - \mathscr{C}^N$ is shorthand for $\boldsymbol{x}-\boldsymbol{y}^{(1)}, \boldsymbol{x}-\boldsymbol{y}^{(2)}, \ldots \boldsymbol{x}-\boldsymbol{y}^{(N)}$. For more general pressure distributions, while the fluid–particles interaction may still be represented by the first term in the right-hand side of (7.2), the effect of the driving pressure field would not be adequately expressed by the second term.

Since the first term does not depend on the configuration and the second one is translationally invariant, upon averaging (7.4) one finds a result of the form

$$\langle p_C \rangle = -\boldsymbol{G}\cdot\boldsymbol{x} + F(t), \tag{7.5}$$

with $F(t)$ independent of position. With (6.6) and (6.4) we can then write the momentum equations (6.1), (6.2) as

$$\rho_C \frac{\partial}{\partial t}\langle \boldsymbol{u}_C \rangle - \boldsymbol{G} = \tfrac{1}{2}\beta_D \rho_C C \frac{\partial}{\partial t}(\bar{\boldsymbol{w}} - \langle \boldsymbol{u}_C \rangle), \tag{7.6}$$

$$\rho_D \frac{\partial \bar{\boldsymbol{w}}}{\partial t} - \boldsymbol{G} = \tfrac{1}{2}\beta_C \rho_C C \frac{\partial}{\partial t}(\langle \boldsymbol{u}_C \rangle - \bar{\boldsymbol{w}}), \tag{7.7}$$

from which

$$\frac{\partial}{\partial t}\langle \boldsymbol{u}_C \rangle = \frac{C\rho_C + 2\rho_D}{\beta_C C\rho_C + (2+\beta_D C)\rho_D \rho_C}\boldsymbol{G}, \tag{7.8}$$

$$\frac{\partial \bar{\boldsymbol{w}}}{\partial t} = \frac{C+2}{\beta_C C\rho_C + (2+\beta_D C)\rho_D}\boldsymbol{G}. \tag{7.9}$$

The form (7.2) implies an averaged particle equation of motion given by

$$v\rho_D \frac{\partial \bar{\boldsymbol{w}}}{\partial t} = \langle \boldsymbol{F} \rangle. \tag{7.10}$$

Upon substitution of (7.8) and (7.9) into the right-hand side of this equation we find

$$\frac{\partial \bar{\boldsymbol{w}}}{\partial t} = \frac{2 + (1-\beta_D) C_a - 2\beta_D}{\beta_C C\rho_C + (2+\beta_D C)\rho_D}\boldsymbol{G}, \tag{7.11}$$

which is only equal to (7.9) if

$$C = (1-\beta_D) C_a - 2\beta_D, \quad C_a = \frac{C + 2\beta_D}{1-\beta_D}. \tag{7.12}$$

It would seem that one might derive this relation by eliminating $\nabla \langle p_C \rangle$ between (6.1), (6.2) and requiring the result to be identically equal to (7.10). This procedure would be incorrect, however (as signalled by the fact that the relation between C_a and C found in this way would depend on $\partial \langle \boldsymbol{u}_C \rangle / \partial t$) because, as noted before, (7.2) can be considered an adequate representation of the disperse-phase force only for a spatially uniform imposed pressure gradient.

Van Wijngaarden (1976) and Kok (1989) calculated C_a in the dilute limit and found, respectively,

$$C_a = 1 + 2.76\beta_D + O(\beta_D^2), \quad C_a = 1 + 3.32\beta_D + O(\beta_D^2). \tag{7.13}$$

Biesheuvel & Spoelstra (1989) showed that the difference between these two dilute-limit results was due to assumptions implicitly made on the final state of the dispersion after the action of the impulsive forces that are assumed to generate the motion. In van Wijngaarden's work, the final particle velocity distribution was the result of the dynamics occurring during the acceleration. In Kok's and Biesheuvel & Spoelstra's approach, on the other hand, it is assumed that the final velocity distribution is uniform. As shown in Sangani *et al.* (1991), other values of the added-mass coefficient

can be obtained with different velocity distributions. Zuber's (1964) well-known result for C_a is

$$C_a = \frac{1+2\beta_D}{1-\beta_D}. \tag{7.14}$$

Upon substitution into (7.12) of the two expressions (7.13) for C_a one finds the dilute-limit results

$$C \approx 1-0.24\beta_D+O(\beta_D^2), \quad C \approx 1+0.32\beta_D+O(\beta_D^2), \tag{7.15}$$

while Zuber's expression simply gives $C = 1$.

At finite volume fractions, it was found in Sangani *et al.* (1991) that Zuber's expression (7.14) fits quite well the numerical results irrespective of ρ_D/ρ_C up to volume fractions as large as 50%. Upon comparison of (7.14) with (7.12), we thus expect that the new coefficient C that we have introduced will be very nearly constant and equal to 1. This expectation is borne out by the computational results of §9.

8. Computational implementation

Equation (7.12) expresses a connection between the coefficient C introduced in (6.4) and the coefficient C_a studied in Sangani *et al.* (1991). The numerical values of C could therefore be directly obtained from that study. Nevertheless, we have recalculated them so as to check directly the validity of the relation (7.12) and also to obtain some further statistical information on the convergence of the averaging process.

The details of the method can be found in Sangani *et al.* (1991). We arrange N_s spheres in a regular array in a cubic cell and then subject each one of them to several thousands of random displacements. A pseudo-random infinite mixture is generated by filling the whole space with copies of this fundamental random cell. The collection of many such realizations constitutes the statistical ensemble. We take advantage of the spatial uniformity prevailing on average to simplify the calculations and to accelerate convergence. Rather than fixing a point $\boldsymbol{x}$ and averaging over the values taken by the fields at that position in the various realizations of the mixture, for each realization, we first calculate volume averages over the fundamental cell and then average these values over the different realizations. It is the result of this combined average that we identify with the ensemble average $\langle\ \rangle$ used in the previous sections.

The pressure field is taken to have the form (7.4) with the part $p_C^N(\boldsymbol{x}-\mathscr{C}^N; t)$ due to the response of the bubbles to the mean gradient $\boldsymbol{G}$ calculated by a singularity method. With the pressure determined, from (3.9) (without the collision force, which is irrelevant for linear motion) one can calculate $\dot{\boldsymbol{w}}$ for each particle in the unit cell and its mean for each configuration

$$\tilde{\dot{\boldsymbol{w}}} = \frac{1}{N_s}\sum_{\alpha=1}^{N_s} \dot{\boldsymbol{w}}^{(\alpha)}. \tag{8.1}$$

The volume-averaged fluid acceleration $\tilde{\dot{\boldsymbol{u}}}_C$ is obtained directly from a knowledge of $\tilde{\dot{\boldsymbol{w}}}$ noting that

$$\begin{aligned}\rho_C \tilde{\dot{\boldsymbol{u}}}_C &= \frac{1}{V_C}\int_{V_C} -\nabla p_C \,\mathrm{d}V_C \\ &= -\frac{1}{V\beta_C}\left[-\boldsymbol{G}V+\int_{\partial V} p_C^N \boldsymbol{n}\,\mathrm{d}S-\sum_{\alpha=1}^{N_s}\int_{S^\alpha} p_C \boldsymbol{n}^\alpha \,\mathrm{d}S^\alpha\right] \\ &= \frac{1}{\beta_C}\boldsymbol{G}-\frac{\beta_D}{\beta_C}\rho_D \tilde{\dot{\boldsymbol{w}}}.\end{aligned} \tag{8.2}$$

Here V_C is the part of the fundamental cell V occupied by the continuous phase. The first step follows from an application of the (generalized) divergence theorem to the domain of integration. The integral of p_C^N over the boundary ∂V of the fundamental cell vanishes by periodicity, and the integral of p_C over the surface of the particles (with the normal directed out of the particles) is just the particle mass times its acceleration. In all realizations the volume fractions are the same so that, identifying N_s/V with the particle number density n, we may write $mN_s/V = \rho_D vn = \rho_D \beta_D$.

Upon averaging the previous result over the realizations of the ensemble we find

$$\rho_C \langle \dot{\boldsymbol{u}}_C \rangle = \frac{1}{\beta_C} \boldsymbol{G} - \frac{\beta_D}{\beta_C} \rho_D \bar{\dot{\boldsymbol{w}}}. \tag{8.3}$$

It is readily verified using this result that the left-hand sides of (7.6) and (7.7) differ by a sign, which confirms the relation $\beta_C \boldsymbol{A}_C + \beta_D \boldsymbol{A}_D = 0$ for this case. From either one of these relations, therefore, the coefficient C can be calculated from $\bar{\boldsymbol{w}}$ obtained as specified above.

Here the relation $\beta_C \boldsymbol{A}_C + \beta_D \boldsymbol{A}_D = 0$ holds not only in an ensemble-average sense, but also for each configuration in the sense of volume averaging. To show this we start by noting that, while the mixture is isotropic on average, each individual realization is not isotropic so that the volume-average analogue of C is a tensor. We therefore rewrite (7.6), (7.7) in the form

$$\rho_C \tilde{\boldsymbol{u}}_C - \boldsymbol{G} = \tfrac{1}{2} \beta_D \rho_C \boldsymbol{C} \cdot (\tilde{\boldsymbol{w}} - \tilde{\boldsymbol{u}}_C), \tag{8.4}$$

$$\rho_D \tilde{\boldsymbol{w}} - \boldsymbol{G} = \tfrac{1}{2} \beta_C \rho_C \boldsymbol{C} \cdot (\tilde{\boldsymbol{u}}_C - \tilde{\boldsymbol{w}}). \tag{8.5}$$

These equations are readily proven to be identical by virtue of (8.2). The numerical evidence indicates that, for each configuration, the tensor $\boldsymbol{C}$ is symmetric.

In the next section we discuss the results derived from the numerical implementation of the approach outlined here.

9. Numerical results

We show in figure 1 a graph of C versus the disperse-phase volume fraction β_D for different values of the density ratio ρ_D/ρ_C. The horizontal dashed line is Zuber's result $C = 1$. The two straight solid lines issuing from $C = 1$, $\beta_D = 0$ are the dilute-limit results (7.15) for $\rho_D/\rho_C = 0$ and infinity, respectively. The range of variation of C is remarkably limited. For any density ratio, it remains within 10% of 1 for volume fractions as large as 30%. For $\rho_D \leqslant \rho_C$ this range extends to over 50%. The dependence on concentration is somewhat stronger when the particle density is much greater than the fluid density, but the maximum difference is about 20% all the way up to nearly close-packing.

To obtain these results, for each volume fraction, we used 40 realizations of 32 particles in the fundamental cell. To assess the randomness of the configurations we checked the pair distribution function and found a very good agreement with the exact solution of the Percus–Yevick equation for hard spheres. For each realization we have taken the imposed pressure gradient $\boldsymbol{G}$ in three mutually orthogonal directions, which effectively triples the number of different realizations of the mixture. At the lower volume fractions (up to 30%), where the interparticle distance tends to be large, we used from three to five singularities per particle (see Sangani *et al.* 1991 for details). At larger volume fractions seven singularities per particle were used. Our study of the

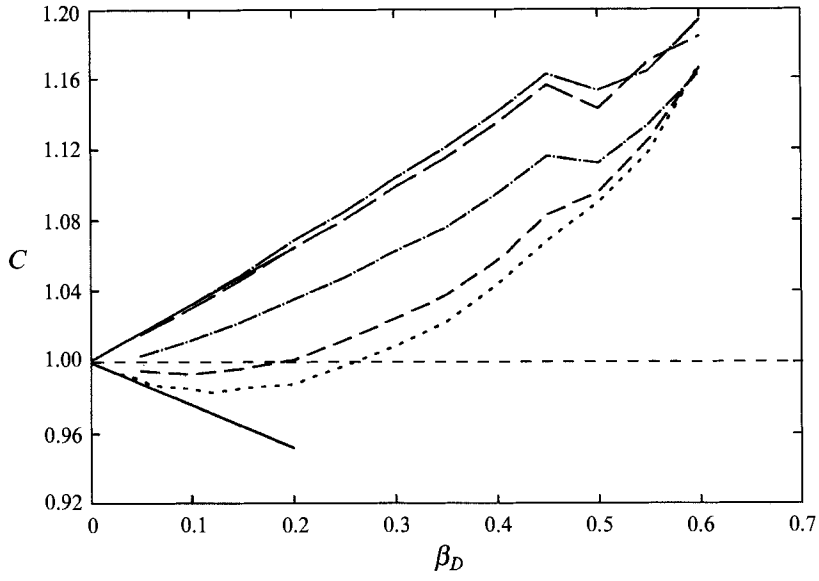

FIGURE 1. The coefficient C defined by (6.4) as a function of the disperse-phase volume fraction β_D for different values of the ratio ρ_D/ρ_C of the disperse- to the continuous-phase density. The two straight lines issuing from $C = 1$, $\beta_D = 0$ are the dilute-limit results (7.15) for $\rho_D/\rho_C = 0$ and infinity, respectively. The other lines correspond, in ascending order, to $\rho_D/\rho_C = 0$, 0.1, 1, 10, 100.

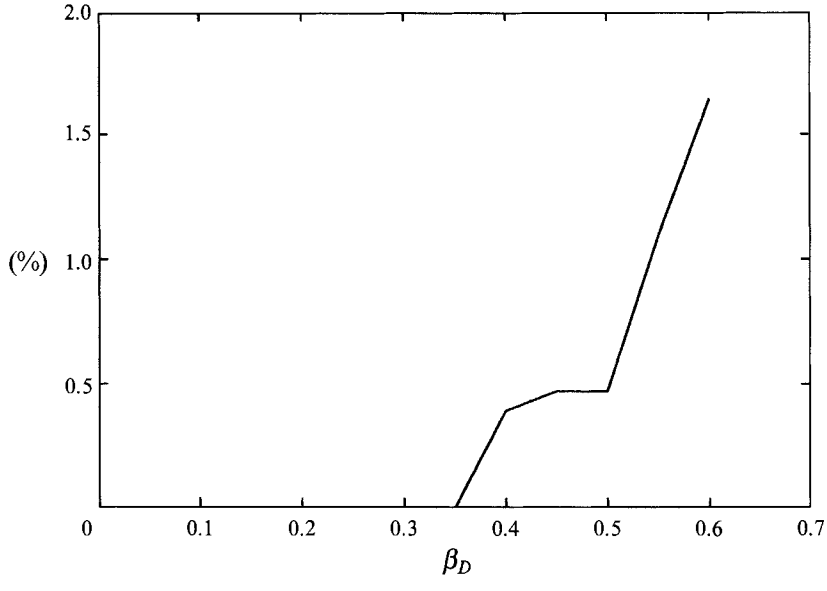

FIGURE 2. Percent of spheres separated by less than 0.0004 times the particle radius from neighbouring spheres as a function of the disperse-phase volume fraction β_D. These results have been obtained from the configurations used to generated the previous figure.

reliability of this numerical method shows that these values are quite sufficient for accuracy and reproducibility well within 1%.

A rather striking aspect of figure 1 is the marked change of slope occurring around $\beta_D = 0.45$. This feature is not a numerical artifact but is in fact quite robust and it most likely corresponds to the disorder-to-order phase transition for hard spheres that is

known to take place in this region (Hansen & McDonald 1986). To test the nature of this phase change we have calculated, as a function of the volume fraction β_D, the fraction of spheres separated by less than ϵa, for different values of ϵ. With the configurations used, this curve is rather irregular unless ϵ is very small. Results for $\epsilon = 0.0004$ (i.e. a separation of 4×10^{-4} times the radius) are shown in figure 2. Although the number of pairs of particles so close together is small, it is seen to undergo a transition and to increase very markedly for β_D between 0.4 and 0.5. The peculiar structure of the curves in figure 1 may therefore be due to the formation of nearly touching sphere clusters in this range, although this is at this stage only a hypothesis. In our earlier paper (Sangani *et al.* 1991) the quantity plotted was C_a rather than C and the different scale necessary to accommodate the greater range of variation obscured this feature.

The results shown in figure 1 are averages. Information on the range of variation of C would also be valuable. This cannot be obtained for C directly, however, because only one value of this quantity is determined from the ensemble. However, we can study this question by calculating the volume-average coefficient defined in the last section by (8.4) or (8.5). This coefficient is a second-rank tensor whose nine components can be calculated from the nine equations generated by taking the vector $\boldsymbol{G}$ in three mutually orthogonal directions as explained above. The result is symmetric and the off-diagonal terms are usually small, of the order of 2% of the diagonal ones. This latter circumstance indicates that the individual realizations are nearly isotropic, which in turn shows that the 32 particles used in the fundamental cell are sufficient. From each calculated tensor we generate three scalars by evaluating the eigenvalues. Since the calculation of the volume-averaged and the ensemble-average C values are different, it is not surprising that the mean over the configurations of the volume-averaged C differs slightly from the ensemble-average C. For example, for $\beta_D = 10\%$, $\rho_D = 0$, we find $C = 0.9875$ for the former and $C = 0.9852$ for the latter. For $\beta_D = 40\%$, $\rho_D = 0$ the two values are 1.047 and 1.044, and for $\beta_D = 40\%$, $\rho_D/\rho_C = 10$ they are 1.134 and 1.133.

Some examples of the probability density distributions of the volume-averaged C are shown as histograms in figure 3. Figure 3(*a*) is for $\beta_D = 10\%$ and $\rho_D = 0$. Figures 3(*b*) and 3(*c*) are both for $\beta_D = 40\%$, one with $\rho_D = 0$ and the other one for $\rho_D/\rho_C = 10$. In these figures the dashed vertical line is the ensemble-average result and the dotted vertical line the mean of the volume-averaged values. An obvious and perhaps disappointing conclusion is that the probability distribution appears rather flat and ranges over $\pm 10\%$ or so from the mean. If the corresponding error bars were plotted in figure 1, hardly any dependence of C on density and volume fraction would be visible. This feature implies that attempts to measure or calculate C with great accuracy are to some extent futile. In any single experiment there would be a substantial margin of uncertainty as to what value effectively prevails. Another use of these results would be to bracket computationally the expected range of experimental data.

To generate a better statistics, these results have been obtained with 200 configurations for $\beta_D = 10\%$ and 100 configurations for $\beta_D = 40\%$ (equivalent to 600 and 300 respectively with the three different choices of $\boldsymbol{G}$). The values of the ensemble-average C for these three cases, 0.9852, 1.044, and 1.133, compare quite well with the corresponding ones, 0.9848, 1.045, and 1.135, obtained with 40 configurations. We thus conclude that the number of configurations used to generate figure 1 is sufficient.

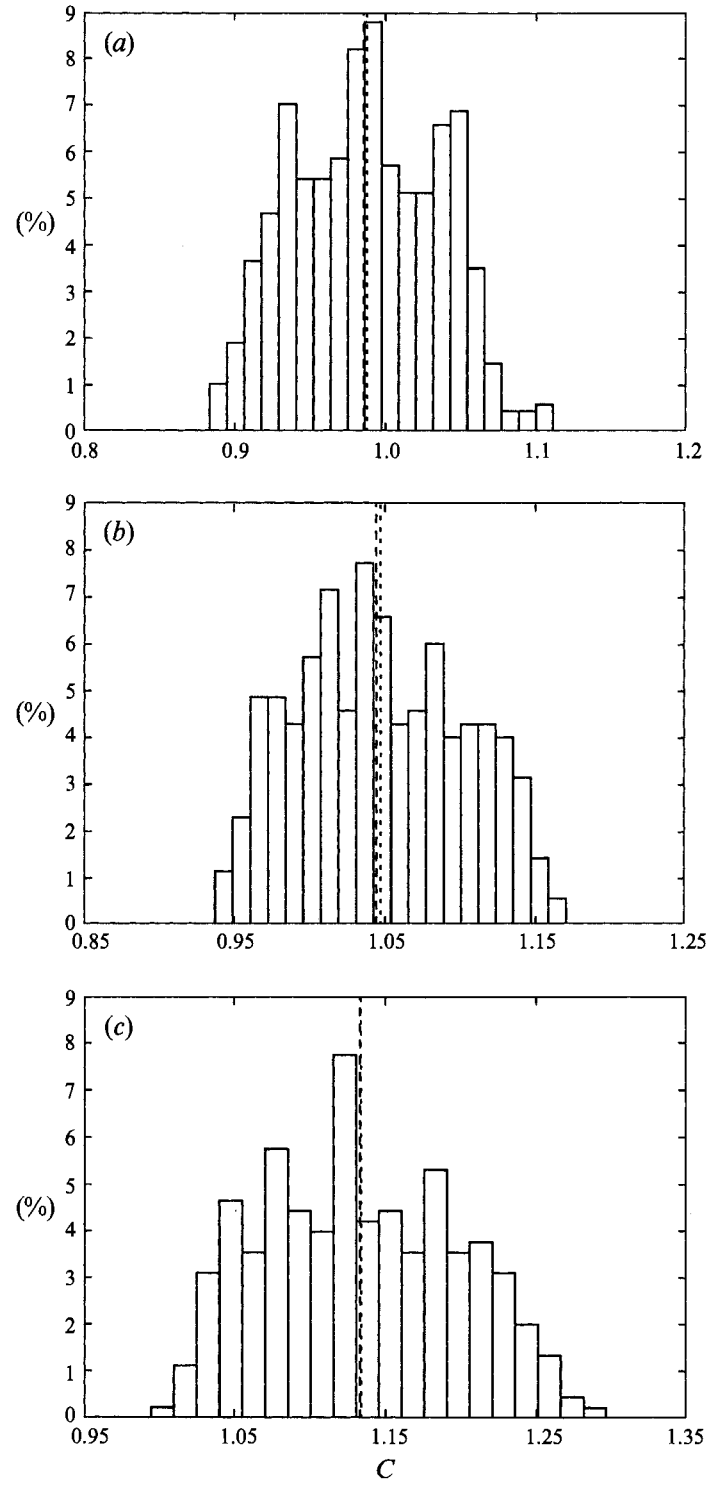

FIGURE 3. Statistical distribution of the volume-averaged values of C (cf. section 9) for (*a*) $\beta_D = 0.1$, $\rho_D = 0$ obtained with 600 configurations, (*b*) $\beta_D = 0.4$, $\rho_D = 0$ obtained with 300 configurations, and (*c*) $\beta_D = 0.4$, $\rho_D/\rho_C = 10$ obtained with 300 configurations. The dashed vertical line is the ensemble-average value and the dotted vertical line the mean of the volume-averaged values.

10. Discussion

Wallis (1991*a*, *b*) has devoted considerable effort to the inertial coupling of spheres in inviscid, locally irrotational flow. Upon comparing equations (74) and (77) of Wallis (1991*b*) with our equations (3.4) and (3.14) (the latter written using the small-particle results of §4) we find

$$-\rho_C \nabla\cdot(\beta_C \boldsymbol{M}_C)-\beta_C \boldsymbol{A}_C = E\rho_C\beta_C\left[\frac{\partial \boldsymbol{u}_C}{\partial t}+\boldsymbol{u}_C\cdot\nabla\boldsymbol{u}_C-\frac{\partial \boldsymbol{u}_D}{\partial t}-\boldsymbol{u}_D\cdot\nabla\boldsymbol{u}_D\right.$$

$$\left.+\boldsymbol{W}\times(\nabla\times\boldsymbol{u}_C)-\boldsymbol{W}\times(\nabla\times\boldsymbol{W})\right]-\beta_C^2\rho_C\frac{\mathrm{d}E}{\mathrm{d}\beta_C}\nabla\cdot\boldsymbol{u}_C+\tfrac{1}{2}W^2\rho_C\beta_C\frac{\mathrm{d}E}{\mathrm{d}\beta_C}\nabla\beta_D, \quad (10.1)$$

$$\nabla\cdot(\beta_D \boldsymbol{T}_C+\rho_C\beta_C\boldsymbol{M}_C) = -\nabla\cdot(\beta_C\rho_C E\boldsymbol{W}\boldsymbol{W})+\nabla[\beta_D(p_D-p_C)]. \quad (10.2)$$

Here we have dropped the angle-bracket notation to indicate averages and $\boldsymbol{W}=\boldsymbol{u}_C-\boldsymbol{u}_D$. The quantities E and p_D are the exertia coefficient and disperse-phase pressure introduced by Wallis. Up to a divergence-free field (that is presumably zero) the second relation implies

$$\beta_D \boldsymbol{T}_C+\rho_C\beta_C\boldsymbol{M}_C = -\beta_C\rho_C E\boldsymbol{W}\boldsymbol{W}+\beta_D(p_D-p_C)\boldsymbol{I}, \quad (10.3)$$

where $\boldsymbol{I}$ is the identity tensor. For the difference p_D-p_C Wallis quotes the relation

$$p_D-p_C = \tfrac{1}{2}\rho_C\beta_C\frac{\mathrm{d}E}{\mathrm{d}\beta_C}W^2, \quad (10.4)$$

concerning which he, however, notes the lack a proof 'that is valid for unsteady motion'. In the dilute limit, with $E=\frac{1}{2}\beta_D$ and (10.4), by the dilute results (5.6) and (5.7) for $\boldsymbol{T}_C$ and $\boldsymbol{M}_C$, the two terms in the left-hand side of (10.3) combine in such a way as to give the right-hand side provided the disperse-phase Reynolds stress $\boldsymbol{M}_D$ can be ignored. Hence our result coincides with Wallis's in this limit. As for (10.1), again in the dilute limit, we find that the two sides of the equation are identical for $\boldsymbol{M}_D=0$. At finite volume fractions, for the linear case of §6, the two models are also equal provided that

$$E=\tfrac{1}{2}\beta_D C. \quad (10.5)$$

In view of this relation, the results presented in the previous section constitute an explicit evaluation of E for this case.

In spite of these similarities, Wallis's work is quite different from ours. The technique that we have developed and used in this paper is quite general and can be applied to a diversity of problems such as Stokes flow, heat conduction, stresses in composites, and others (Zhang 1993; Prosperetti & Zhang 1994). Wallis's approach, on the other hand, rests heavily on potential theory. Secondly, his results depend on a number of assumptions which, though physically insightful, lack a rigorous justification. The validity of (10.4) is one example. Others are his reliance on the dipole approximation, and his use of area and volume, as opposed to ensemble, averaging. Although all these averaging techniques coincide for homogeneous systems, care is needed in interpreting spatial averages for dense, non-homogeneous mixtures whereas ensemble averages are always well defined. Finally, his results presuppose that all particles move with the same velocity.

Sangani & Didwania (1993*a*) have carried out a study similar to ours for $\rho_D=0$ neglecting contributions from the particles' Reynolds stress. Rather than obtaining separate equations for the continuous and disperse phases, they obtain a mixture

equation and a disperse-phase equation. The present mixture equation can be obtained by adding the two momentum equations (4.11), (4.12) to find

$$\rho_C \frac{\partial}{\partial t}(\beta_C \boldsymbol{u}_C) + \rho_C \nabla\cdot(\beta_C \boldsymbol{u}_C \boldsymbol{u}_C) + \nabla p_C$$
$$= \nabla\cdot(\beta_C \rho_C \boldsymbol{M}_C + \beta_D \boldsymbol{T}_C + \beta_D \rho_D \boldsymbol{M}_D) + (\beta_C \rho_C + \beta_D \rho_D)\boldsymbol{g} - \langle \boldsymbol{F}_{cL}\rangle + n\bar{\boldsymbol{F}}_c. \quad (10.6)$$

The left-hand side of this equation is the same as that of Sangani & Didwania. After carrying out a small-particle approximation similar to that of §4, Sangani & Didwania write the right-hand side as $\nabla(P\boldsymbol{I} + \beta_D \boldsymbol{\Sigma})$, where $P = \beta_C p_C + \beta_D \rho_D$ is the total mixture pressure that may be identically rewritten as $P = p_C - \beta_D(p_C - p_D)$. One of the terms in $\boldsymbol{\Sigma}$ is identical to our continuous-phase Reynolds stress tensor $\boldsymbol{M}_C$. The remaining terms, with the neglect of viscosity, are the divergence of

$$P\delta_{ij} + \beta_D \Sigma^S_{ij} = [p_C - \beta_D(p_C - p_D)]\,\delta_{ij} + \frac{3\beta_D}{4\pi a^2}\int_{r=a} \langle p_C - p_D\rangle_1 n_i n_j \,\mathrm{d}S_r$$
$$= p_C\,\delta_{ij} + \frac{3\beta_D}{4\pi a^2}\int_{r=a} [\langle p_C - p_D\rangle_1 - (p_C - p_D)]\, n_i n_j \,\mathrm{d}S_r. \quad (10.7)$$

With the approximation $\langle p_D\rangle_1 \approx p_D$, this integral is identical to our $-\boldsymbol{T}_C$, so that Sangani & Didwania's term $P\delta_{ij} + \beta_D \boldsymbol{\Sigma}$ equals our $p_C\,\delta_{ij} - \beta_D \boldsymbol{T}_C - \rho_C \beta_C \boldsymbol{M}_C$. The correspondence is close since, as demonstrated by the explicit results of §5, both $\beta_D \boldsymbol{T}_C$ and $\boldsymbol{M}_C$ are proportional to β_D for small β_D. A comparison of Sangani & Didwania's other momentum equation with the present results is given in Appendix C, and a more complete discussion can be found in Bulthuis, Prosperetti & Sangani (1994).

In several papers, Drew, Lahey and co-workers (Drew 1989; Arnold, Drew & Lahey 1989; Park, Drew & Lahey 1992) have presented averaged equations derived by a version of the so-called 'cell model'. After correction of what appear to be misprints, their results agree with those that we presented for the dilute case in §5. There are some important differences, however. The first one is that their final expressions are given in expanded form, failing to recognize that they are the divergence of the tensor $\boldsymbol{T}_C$ defined in (4.10). As a consequence, the derivation of the equations is also more complicated. Secondly, their result is based on a rather *ad hoc* implementation of ensemble averaging. Thirdly, our derivation explicitly shows the result to be only valid to $O(\beta_D)$ rather than for arbitrary volume fractions. Finally, they take the particles to move with the same velocity so that their equations do not contain a term comparable to $\boldsymbol{M}_D$.

The trace of the divergence term in the right-hand side of the continuous-phase momentum equation (5.11) is

$$\tfrac{1}{4}\rho_C \nabla[\beta_D(\boldsymbol{u}_C - \boldsymbol{u}_D)^2] = \tfrac{1}{4}(\boldsymbol{u}_C - \boldsymbol{u}_D)^2 \nabla\beta_D + \tfrac{1}{2}\beta_D(\boldsymbol{u}_C - \boldsymbol{u}_D)\cdot\nabla(\boldsymbol{u}_C - \boldsymbol{u}_D). \quad (10.8)$$

The first term in the right-hand side of this equation is present in several multiphase flow models (see e.g. Banerjee & Chan 1982; Pauchon & Banerjee 1986; Pauchon & Smereka 1992) where it is written as $(p_C - p_S)\nabla\beta_D$, with p_S the surface average pressure. The expression $p_C - p_S = \frac{1}{4}\rho_C(\boldsymbol{u}_C - \boldsymbol{u}_D)^2$ (cf. (10.4)) is deduced by averaging the pressure over the surface of an isolated sphere in potential flow. It has been argued in Prosperetti & Jones (1984) that the correct form of this contribution should instead be $\nabla[\beta_D(p_C - p_S)]$. While this suggestion is supported by the present results, the tensorial nature of this term was not identified in that study.

A distinctive feature of our work has been the manner in which the disperse-phase momentum equation has been derived without explicit use of the particle constitutive

relation. This approach, which is made possible by using averages separately defined for each phase, may be preferable to the more common one in which the averaged quantities are required to be defined at every point in space (see e.g. Batchelor 1970, 1972, 1974). Indeed, the continuous-phase averaged momentum equation cannot physically be expected to depend on the detailed internal mechanics of the particles as long as their outer boundary appears rigid to the suspending phase. We also find that the motion of the disperse phase is determined by the continuous-phase pressure, rather than by some 'average' pressure that includes a contribution from the particles as, e.g., in Biesheuvel & van Wijngaarden (1984), Wallis (1991*a*, *b*), and many others. Again, this conforms with physical intuition. Any net effect of the particle internal pressure on the motion of the disperse phase can only arise through a coupling with the continuous-phase pressure mediated by a stress-balance condition at the particle–fluid interface. The approach that we have taken avoids this rather involute conceptual description and deals with the continuous-phase pressure directly.

With $\boldsymbol{M}_D = 0$ the dilute-limit model derived in §5 is not hyperbolic and, therefore (Jones & Prosperetti 1985), unstable. As shown in §4, the model is invalid at the short spatial scales that must be considered to test hyperbolicity and the first conclusion is therefore of little significance *per se*. The instability, on the other hand, is real and reflects the instability of the initial-value problem for spheres in potential flow demonstrated by the direct numerical simulations of Sangani & Didwania (1993*b*) and by the considerations of Smereka & Milton (1991). The particle fluctuation velocity (which has been ignored by virtually all previous investigators) may be expected to counter somewhat the mutual attractive force between the particles and hence mitigate the instability.

The numerical results of §9 for concentrated mixtures show that our equivalent of the added-mass coefficient depends very little on volume fraction and density. We have studied its statistics and have found that a $\pm 10\%$ variability can be expected between individual realizations of the ensemble. This result indicates that, for practical purposes, little would be gained by attempting to determine the ensemble-average value of this quantity with greater precision and a value $C = 1$ can be assumed for practical purposes.

These computations furnish one example of a manner in which our results are useful in practice. For example, on the basis of (5.6), one may try to parametrize the form of $\boldsymbol{T}_C$ at finite volume fractions as

$$\begin{aligned}\boldsymbol{T}_C = {} & \tfrac{1}{5}\rho_C[2G_1(\langle \boldsymbol{u}_C\rangle - \langle \boldsymbol{u}_D\rangle)^2 \boldsymbol{I} \\ & -\tfrac{9}{4}G_2(\langle \boldsymbol{u}_C\rangle - \langle \boldsymbol{u}_D\rangle)(\langle \boldsymbol{u}_C\rangle - \langle \boldsymbol{u}_D\rangle)] - \tfrac{1}{5}\rho_C[2G_1(\mathrm{Tr}\,\boldsymbol{M}_D)\boldsymbol{I} - \tfrac{9}{4}G_2\boldsymbol{M}_D], \quad (10.9)\end{aligned}$$

where the coefficients G_1 and G_2 are functions of β_D and ρ_D/ρ_C that equal 1 for $\beta_D = 0$, and to obtain these quantities numerically by means of direct simulations. In a broader perspective, the quantities that are needed to close the model, such as $\boldsymbol{T}_C$ or $\boldsymbol{A}_D$, are expressed in terms of well-defined computable integrals (cf. (4.10) and (3.13)) for which direct numerical simulation can be quite powerful.

Two important effects – particle collision and viscosity – have been neglected in this study. The first one is small for the cases we have explicitly considered – dilute suspensions or small-amplitude motion – but will be significant in many other problems. Drag forces, as shown in Sangani & Didwania (1993*b*), are essential to describe flows in which the particle distribution function cannot be prescribed as here but evolves with time. These effects will be considered in future applications of the present approach.

Finally, we note that the same technique demonstrated here can be applied to the

case in which the particles have more degrees of freedom simply by including them as additional arguments of the probability density function $P(N;t)$. In a separate paper (Zhang & Prosperetti 1994) we study the case of particles with a variable radius. The extension to complex shapes characterized by orientation etc. is immediate. By the same technique we have also obtained dilute-limit results for Stokes flow, heat conduction and convection, and thermocapillary flow (Zhang 1993; Prosperetti & Zhang 1993).

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Appendix A

We derive here an equation corresponding to (2.31) for the partial time derivative of the conditional average $\langle f_C\rangle_K$. There is a subtle point here that may be explained as follows. At time t, $\langle f_C\rangle_K$ is calculated by averaging f_C over the subset of all realizations such that K particles are in some prescribed configuration $\mathscr{C}^K$. At time $t+\mathrm{d}t$, for each one of these realizations, the configuration of those K particles will have changed in a different way since it is affected also by the motion of the $N-K$ particles that are not tracked in detail. Thus, if one were to think of $\partial\langle f_C\rangle_K/\partial t$ as the limit of the ratio of the increments, there is no straightforward rule on which configurations to include in calculating $\langle f_C\rangle_K$ at time $t+\mathrm{d}t$. In the presence of this ambiguity, which definition to adopt for this derivative depends on the ultimate use of this quantity. If, for example, one needs to average the liquid momentum equation subject to one particle being in a certain position at time t, where the particle in question is at $t+\mathrm{d}t$ is irrelevant and the proper average must include all the possibilities. Because of these considerations we define the partial time derivative of $\langle f_C\rangle_K$ at $\boldsymbol{x}$ fixed by

$$\frac{\partial}{\partial t}(\beta_C^K\langle f_C\rangle_K)=\frac{1}{(N-K)!}\int \mathrm{d}\mathscr{C}^{N-K}\left\{\chi_C\left(\frac{\hat{\partial} f_C}{\hat{\partial} t}P(N-K\,|\,K;t)+f_C\frac{\partial P(N-K\,|\,K;t)}{\partial t}\right)\right.$$
$$\left.+\sum_{\alpha=1}^{K}\left[\boldsymbol{w}^\alpha\cdot\boldsymbol{\nabla}_\alpha(\chi_C f_C P(N-K\,|\,K;t))+\dot{\boldsymbol{w}}^\alpha\cdot\boldsymbol{\Delta}_\alpha(\chi_C f_C P(N-K\,|\,K;t))\right]\right\}. \quad \text{(A 1)}$$

The derivation of an explicit result requires an evolution equation for the conditional probability. From the definition $P(N-K\,|\,K;t)=P(N;t)/P(K;t)$ this can be obtained from the probability equation for $P(N;t)$ and the corresponding equation for $P(K;t)$. The latter is readily obtained by averaging the probability equation (2.2) over the configurations of the late $N-K$ particles. The result is

$$\frac{\partial P(K;t)}{\partial t}+\sum_{\alpha=1}^{K}\{\boldsymbol{\nabla}_\alpha\cdot[\boldsymbol{w}^\alpha P(K;t)]+\boldsymbol{\Delta}_\alpha\cdot[\langle\!\langle\dot{\boldsymbol{w}}^\alpha\rangle\!\rangle_K P(K;t)]\}=0, \quad \text{(A 2)}$$

where, for $\alpha=1,2,\ldots,K$,

$$\langle\!\langle\dot{\boldsymbol{w}}^\alpha\rangle\!\rangle_K=\frac{1}{(N-K)!}\int P(N-K\,|\,K;t)\,\dot{\boldsymbol{w}}^\alpha\,\mathrm{d}\mathscr{C}^{N-K}. \quad \text{(A 3)}$$

The new average introduced here is somewhat reminiscent of that defined in (2.37) but bears no simple relation to it and is best indicated by a different symbol.

With this result, some straightforward manipulations lead to

$$\frac{\partial}{\partial t}(\beta_C^K\langle f_C\rangle_K)=\beta_C^K\left\langle\frac{\partial f_C}{\partial t}\right\rangle_K+\frac{1}{(N-K)!}\sum_{\alpha=1}^{N}\int \mathrm{d}\mathscr{C}^{N-K} f_C\,\boldsymbol{w}^\alpha\cdot\boldsymbol{\nabla}_\alpha\chi_C\,P(N-K|\,K;t)$$
$$+\frac{\beta_C^K}{P(K;t)}\sum_{\alpha=1}^{K}\langle f_C\,\boldsymbol{\Delta}_\alpha\cdot[(\langle\!\langle\dot{\boldsymbol{w}}^\alpha\rangle\!\rangle_K-\dot{\boldsymbol{w}}^\alpha)\,P(K;t)]\rangle_K. \quad \text{(A 4)}$$

The corresponding definition of the spatial derivative is unambiguous and the calculation simpler. The result is

$$\boldsymbol{\nabla}\cdot(\beta_C^K\langle f_C\,\boldsymbol{u}_C\rangle_K)=\beta_C^K\langle\boldsymbol{\nabla}\cdot(f_C\,\boldsymbol{u}_C)\rangle_K+\frac{1}{(N-K)!}\int \mathrm{d}\mathscr{C}^{N-K} f_C\,\boldsymbol{u}_C\cdot\boldsymbol{\nabla}\chi_C\,P(N-K|\,K;t). \quad \text{(A 5)}$$

Again using the kinematic boundary condition (2.30), the analogue of the result (2.31) for the conditional average is found to be

$$\frac{\partial}{\partial t}(\beta_C^K\langle f_C\rangle_K)+\boldsymbol{\nabla}\cdot(\beta_C^K\langle f_C\,\boldsymbol{u}_C\rangle_K)=\beta_C^K\left\langle\frac{\partial f_C}{\partial t}+\boldsymbol{\nabla}\cdot(f_C\,\boldsymbol{u}_C)\right\rangle_K$$
$$+\frac{\beta_C^K}{P(K;t)}\sum_{\alpha=1}^{K}\langle f_C\,\boldsymbol{\Delta}_\alpha\cdot[(\dot{\boldsymbol{w}}^\alpha-\langle\!\langle\dot{\boldsymbol{w}}^\alpha\rangle\!\rangle_K)\,P(K;t)]\rangle_K. \quad \text{(A 6)}$$

With this result we may now write the average of the continuous-phase continuity and momentum equations conditional to one particle being fixed at $\boldsymbol{y}$ with centre-of-mass velocity $\boldsymbol{w}$. By choosing $f_C=1$ we find

$$\frac{\partial\beta_C^1}{\partial t}+\boldsymbol{\nabla}\cdot(\beta_C^1\langle\boldsymbol{u}_C\rangle_1)=\frac{\beta_C^1}{P(1;t)}\langle\boldsymbol{\Delta}_1\cdot[(\dot{\boldsymbol{w}}-\langle\!\langle\dot{\boldsymbol{w}}\rangle\!\rangle_1)\,P(1;t)]\rangle_1, \quad \text{(A 7)}$$

to be compared with the unconditional form (3.3). Similarly, by taking $f_C=\boldsymbol{u}_C$, we find

$$\frac{\partial}{\partial t}(\beta_C^1\langle\boldsymbol{u}_C\rangle_1)+\boldsymbol{\nabla}\cdot(\beta_C^1\langle\boldsymbol{u}_C\,\boldsymbol{u}_C\rangle_1)=-\frac{\beta_C^1}{\rho_C}\langle\boldsymbol{\nabla}p_C\rangle_1+\beta_C^1\boldsymbol{g}$$
$$+\frac{\beta_C^1}{P(1;t)}\langle\boldsymbol{u}_C\,\boldsymbol{\Delta}_1\cdot[(\dot{\boldsymbol{w}}-\langle\!\langle\dot{\boldsymbol{w}}\rangle\!\rangle_1)\,P(1;t)]\rangle_1, \quad \text{(A 8)}$$

analogous to (3.4). The quantity $\dot{\boldsymbol{w}}-\langle\!\langle\dot{\boldsymbol{w}}\rangle\!\rangle_1$ is smaller than $O(1)$ for $\beta_D\to 0$. Similarly, to the same accuracy, $\beta_C^1=1$, $\langle\boldsymbol{u}_C\,\boldsymbol{u}_C\rangle_1=\langle\boldsymbol{u}_C\rangle_1\langle\boldsymbol{u}_C\rangle_1$, $\langle\boldsymbol{\nabla}p_C\rangle_1=\boldsymbol{\nabla}\langle p_C\rangle_1$, while $\partial\beta_C^1/\partial t=-\partial\beta_D^1/\partial t$ is negligible. Using these approximations, (5.1) and (5.2) are obtained.

Appendix B. Detailed derivation of the results of §5

In view of the boundary conditions at infinity (5.3), it is advantageous to consider simultaneously with (5.1), (5.2) the unconditionally averaged equations, also to $O(1)$, namely

$$\boldsymbol{\nabla}\cdot\langle\boldsymbol{u}_C\rangle=0, \quad \text{(B 1)}$$

$$\rho_C\left[\frac{\partial\langle\boldsymbol{u}_C\rangle}{\partial t}+\langle\boldsymbol{u}_C\rangle\cdot\boldsymbol{\nabla}\langle\boldsymbol{u}_C\rangle\right]+\boldsymbol{\nabla}\langle p_C\rangle=0. \quad \text{(B 2)}$$

Furthermore, it is convenient to change the frame of reference to one in which particle 1 is at rest. Indicating by a prime quantities evaluated in this non-inertial frame, we write

$$\langle \boldsymbol{u}'_C \rangle_1 = \boldsymbol{\nabla}\phi'_1, \quad \langle \boldsymbol{u}'_C \rangle = \boldsymbol{\nabla}\phi'. \tag{B 3}$$

The Bernoulli integral of (5.2) is then

$$\frac{\partial \phi'_1}{\partial t'} + \tfrac{1}{2}\langle \boldsymbol{u}'_C \rangle_1 \cdot \langle \boldsymbol{u}'_C \rangle_1 + \dot{\boldsymbol{w}} \cdot \boldsymbol{x} + \frac{\langle p_C \rangle_1}{\rho_C} = \frac{\partial \phi'}{\partial t'} + \tfrac{1}{2}\langle \boldsymbol{u}'_C \rangle \cdot \langle \boldsymbol{u}'_C \rangle + \dot{\boldsymbol{w}} \cdot \boldsymbol{x} + \frac{\langle p_C \rangle}{\rho_C}, \tag{B 4}$$

from which

$$\frac{\langle p_C \rangle_1 - \langle p_C \rangle}{\rho_C} = \frac{\partial}{\partial t'}(\phi' - \phi'_1) + \tfrac{1}{2}(\langle \boldsymbol{u}'_C \rangle \cdot \langle \boldsymbol{u}'_C \rangle - \langle \partial \boldsymbol{u}'_C \rangle_1 \cdot \langle \boldsymbol{u}'_C \rangle_1). \tag{B 5}$$

It should be noted that, since we are only interested in the difference between the two pressures, all the calculations can be carried out in the non-inertial frame with no need to correct for the acceleration of the frame which affects $\langle p_C \rangle$ and $\langle p_C \rangle_1$ equally.

We start by expanding ϕ' around the centre $\boldsymbol{y}$ of the particle to find, at the generic field point $\boldsymbol{x}$,

$$\phi'(\boldsymbol{x}) = \sum_{l=0}^{\infty} \frac{1}{l!} (\boldsymbol{x}-\boldsymbol{y})^{(l)} \boldsymbol{\nabla}^{(l)} \phi'(\boldsymbol{y}). \tag{B 6}$$

Here and in the following we use an abbreviated notation such that

$$(\boldsymbol{x}-\boldsymbol{y})^{(l)} \boldsymbol{\nabla}^{(l)} \phi'(\boldsymbol{y}) \equiv (\boldsymbol{x}-\boldsymbol{y})_{i_1} (\boldsymbol{x}-\boldsymbol{y})_{i_2} \dots (\boldsymbol{x}-\boldsymbol{y})_{i_l} \partial_{i_1} \partial_{i_2} \dots \partial_{i_l} \phi'(\boldsymbol{y}). \tag{B 7}$$

The sphere centred at $\boldsymbol{y}$ is now introduced in the flow described by ϕ'. It will be recalled that the frame of reference has been selected so that this sphere is at rest. To calculate the new potential ϕ'_1 it is useful to note that the quantity

$$\frac{1}{l! r^l} (\boldsymbol{x}-\boldsymbol{y})^{(l)} \boldsymbol{\nabla}^{(l)} \phi'(\boldsymbol{y}) \tag{B 8}$$

is independent of $r = |\boldsymbol{x}-\boldsymbol{y}|$ and only depends on the angular coordinates measured from the centre of the sphere. With this remark, and by either using Weiss's theorem or proceeding directly, it is easy to find

$$\phi'_1 = \sum_{l=0}^{\infty} \frac{1}{l!} \left[1 + \frac{l}{l+1} \left(\frac{a}{r} \right)^{2l+1} \right] (\boldsymbol{x}-\boldsymbol{y})^{(l)} \boldsymbol{\nabla}^{(l)} \phi'(\boldsymbol{y}). \tag{B 9}$$

From this expression the component $\langle u'_{Ci} \rangle_1$ of the velocity $\langle \boldsymbol{u}'_C \rangle_1$ in the i-direction is readily found as

$$\langle u'_{Ci} \rangle_1 = \langle u'_{Ci} \rangle + \sum_{l=1}^{\infty} \frac{l}{(l+1)!} \left(\frac{a}{r} \right)^{2l+1} \left[l\delta_{ik} - (2l+1) \frac{(\boldsymbol{x}-\boldsymbol{y})_i (\boldsymbol{x}-\boldsymbol{y})_k}{r^2} \right] (\boldsymbol{x}-\boldsymbol{y})^{(l-1)} \partial_k \boldsymbol{\nabla}^{(l-1)} \phi'(\boldsymbol{y}), \tag{B 10}$$

where

$$\langle u'_{Ci} \rangle = \sum_{l=1}^{\infty} \frac{1}{(l-1)!} (\boldsymbol{x}-\boldsymbol{y})^{(l-1)} \partial_i \boldsymbol{\nabla}^{(l-1)} \phi'(\boldsymbol{y}), \tag{B 11}$$

is the ith component of the unconditionally averaged velocity $\langle \boldsymbol{u}'_C \rangle$.

B.1. *Calculation of A_D and T_C*

We use (B 4) to express the integrand in the approximation (4.7) for A_D and (B 6) and (B 9) to calculate the time derivative of the difference of the potentials. To lowest order in β_D one finds

$$\int \mathrm{d}S\, \boldsymbol{n} \frac{\partial}{\partial t}(\phi' - \phi_1') = -\tfrac{1}{2} v \frac{\partial \langle \boldsymbol{u}_C' \rangle}{\partial t}$$

$$= -\tfrac{1}{2} v \left(\frac{\partial \langle \boldsymbol{u}_C \rangle}{\partial t} + \boldsymbol{w} \cdot \boldsymbol{\nabla} \langle \boldsymbol{u}_C \rangle - \dot{\boldsymbol{w}} \right), \tag{B 12}$$

where the last step follows upon expressing the result in terms of quantities referred to the original frame.

To calculate the contribution of the nonlinear term in (B 6), we start by evaluating it on the surface of the particle centred at $\boldsymbol{y}$ to find

$$\begin{aligned}
&\langle \boldsymbol{u}_C' \rangle_1 \cdot \langle \boldsymbol{u}_C' \rangle_1 - \langle \boldsymbol{u}_C' \rangle \cdot \langle \boldsymbol{u}_C' \rangle \\
&\quad = \sum_{l=1}^{\infty} \sum_{m=0}^{\infty} \frac{l p a^{l+p-2}}{(l+1)!(p+1)!} [(3pl+p+l)\,\delta_{ij} - (2l+1)(2p+1)\, n_i n_j] \\
&\qquad \times [\boldsymbol{n}^{(l-1)}\, \partial_i \boldsymbol{\nabla}^{(l-1)} \phi'(\boldsymbol{y})]\, [\boldsymbol{n}^{(p-1)}\, \partial_j \boldsymbol{\nabla}^{(p-1)} \phi'(\boldsymbol{y})] \\
&\quad = \tfrac{1}{4}(5 \partial_k \phi' \partial_k \phi' - 9 n_k n_j \partial_k \phi' \partial_j \phi') + a(3 n_i \partial_i \partial_k \phi' \partial_k \phi' \\
&\qquad - 5 n_i n_j n_k \partial_i \partial_k \phi' \partial_j \phi') + O(a^2).
\end{aligned} \tag{B 13}$$

Upon substitution into the surface integral in (4.7) we find

$$\frac{1}{2} \int \mathrm{d}S\, \boldsymbol{n} (\langle \boldsymbol{u}_C' \rangle \cdot \langle \boldsymbol{u}_C' \rangle - \langle \boldsymbol{u}_C' \rangle_1 \cdot \langle \boldsymbol{u}_C' \rangle_1) = \tfrac{1}{2} v (\boldsymbol{w} - \langle \boldsymbol{u}_C \rangle) \cdot \boldsymbol{\nabla} \langle \boldsymbol{u}_C \rangle. \tag{B 14}$$

Combining (B 12) and (B 14) we then have

$$\int \mathrm{d}S\, \boldsymbol{n} (\langle p_C \rangle_1 - \langle p_C \rangle) = -\tfrac{1}{2} \rho_C v \left[\frac{\partial \langle \boldsymbol{u}_C \rangle}{\partial t} + \langle \boldsymbol{u}_C \rangle \cdot \boldsymbol{\nabla} \langle \boldsymbol{u}_C \rangle - \dot{\boldsymbol{w}} \right]. \tag{B 15}$$

The last step is to perform the integration over the particle velocity $\boldsymbol{w}$. Since $\langle \boldsymbol{u}_C \rangle$ is independent of this quantity, by (2.15), (4.1), and (2.38), we have

$$\boldsymbol{A}_D = \tfrac{1}{2} \rho_C \left[\frac{\partial \langle \boldsymbol{u}_C \rangle}{\partial t} + \langle \boldsymbol{u}_C \rangle \cdot \boldsymbol{\nabla} \langle \boldsymbol{u}_C \rangle \right] - \tfrac{1}{2} \rho_C \bar{\dot{\boldsymbol{w}}}. \tag{B 16}$$

The last term $\bar{\dot{\boldsymbol{w}}}$ has been calculated in (3.15) so that

$$\beta_D \boldsymbol{A}_D = -\tfrac{1}{2} \rho_C \beta_D \left\{ \frac{\partial \bar{\boldsymbol{w}}}{\partial t} + \bar{\boldsymbol{w}} \cdot \boldsymbol{\nabla} \bar{\boldsymbol{w}} - \frac{\partial \langle \boldsymbol{u}_C \rangle}{\partial t} - \langle \boldsymbol{u}_C \rangle \cdot \boldsymbol{\nabla} \langle \boldsymbol{u}_C \rangle \right\} + \tfrac{1}{2} \rho_C \boldsymbol{\nabla} \cdot (\beta_D \boldsymbol{M}_D). \tag{B 17}$$

With the substitution of $\langle \boldsymbol{u}_D \rangle$ for $\bar{\boldsymbol{w}}$, which is legitimate to the present order of accuracy, we recover the form (5.5) given in the text.

The calculation of $\boldsymbol{T}_C$ is very similar. The integrand only differs from that of $\boldsymbol{A}_D$ by the presence of an extra factor $\boldsymbol{n}$, which singles out different terms of the expansions (B 6), (B 9), and (B 13). The calculation is otherwise identical to the one just described and gives the result quoted in (5.6) in the text.

B.2. *The Reynolds stresses*

For the continuous phase we have, to the present order,

$$\begin{aligned}
\boldsymbol{M}_C &= -\langle (\boldsymbol{u}_C - \langle \boldsymbol{u}_C \rangle)(\boldsymbol{u}_C - \langle \boldsymbol{u}_C \rangle) \rangle \\
&= -\langle (\langle \boldsymbol{u}_C \rangle_1 - \langle \boldsymbol{u}_C \rangle)(\langle \boldsymbol{u}_C \rangle_1 - \langle \boldsymbol{u}_C \rangle) \rangle_1 + O(\beta_D).
\end{aligned} \tag{B 18}$$

Thus

$$\boldsymbol{M}_C = -\int \mathrm{d}^3 w \int \mathrm{d}^3 y\, P(\boldsymbol{y}, \boldsymbol{w}; t)(\langle \boldsymbol{u}_C \rangle_1 - \langle \boldsymbol{u}_C \rangle)(\langle \boldsymbol{u}_C \rangle_1 - \langle \boldsymbol{u}_C \rangle). \tag{B 19}$$

We use the earlier result (B 10) for $\langle \boldsymbol{u}_C \rangle_1 - \langle \boldsymbol{u}_C \rangle = \langle \boldsymbol{u}'_C \rangle_1 - \langle \boldsymbol{u}'_C \rangle$ which, to leading order, is

$$\langle u_{Ci} \rangle_1 - \langle u_{Ci} \rangle = \frac{1}{2}\left(\frac{a}{r}\right)^3 (\delta_{ij} - 3n_i n_j) \langle u'_{Cj} \rangle (\boldsymbol{y}, t) + O\left(\frac{a^5}{r^5}\right), \tag{B 20}$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$. Upon substitution into (B 19) and change of the integration variable from $\boldsymbol{y}$ to $\boldsymbol{r}$ we have an expression with the structure

$$\boldsymbol{I} = \int_{r \geqslant a} \left(\frac{a}{r}\right)^6 F(\boldsymbol{x}, \boldsymbol{r})\, \mathrm{d}^3 r. \tag{B 21}$$

By adding and subtracting $F(\boldsymbol{x}, \boldsymbol{0})$ in the integrand we have

$$\boldsymbol{I} = \int_{r \geqslant a} \left(\frac{a}{r}\right)^6 F(\boldsymbol{x}, \boldsymbol{0})\, \mathrm{d}^3 r + a^3 \int_{s \geqslant 1} s^{-6}[F(\boldsymbol{x}, a\boldsymbol{s}) - F(\boldsymbol{x}, \boldsymbol{0})]\, \mathrm{d}^3 s, \tag{B 22}$$

where $\boldsymbol{s} = \boldsymbol{r}/a$. It is clear that, as $a \to 0$, the second term tends to zero faster than a^3 and therefore can be neglected to the present accuracy. With this simplification the integral in (B 19) is straightforward and gives the result (5.7) quoted in the text.

For the disperse-phase Reynolds stress see the comments at the end of § 5.

B.3. *Rotating frame*

Here the correction to $\boldsymbol{A}_D$ given in (5.9) is derived.

Let us consider, in a frame rotating with an arbitrary angular velocity $\boldsymbol{\Omega}(t)$, a flow that is potential in the inertial laboratory frame. In the rotating frame the continuous-phase velocity field has a uniform vorticity $\nabla \times \langle \boldsymbol{u}_C \rangle = 2\boldsymbol{\Omega}$ and, since the expression (5.5) has been obtained under the assumption of irrotational flow, it will not represent the correct value of $\boldsymbol{A}_D$ in the rotating frame. To amend this relation for rotation we write

$$\boldsymbol{A}'_D = \boldsymbol{A}_D + \rho_C \boldsymbol{X}, \tag{B 23}$$

where $\boldsymbol{A}_D$ is the form given by (5.5) and $\boldsymbol{X}$ must be determined. To this end we require that, with this correction, $\boldsymbol{A}'_D$ transform in an objective way upon a further change of the coordinate system to another rotating frame, i.e.

$$\boldsymbol{A}'^*_D = \boldsymbol{Q}(t)\, \boldsymbol{A}'_D, \tag{B 24}$$

where $\boldsymbol{Q}$ is a suitable orthogonal matrix (see e.g. Drew *et al.* 1979) and the asterisk denotes values in the second rotating frame. Upon imposing this condition on (B 23), using the standard rules for the transformation of the velocity field, one finds

$$\boldsymbol{X}^* - \dot{\boldsymbol{Q}}(\bar{\boldsymbol{w}} - \langle \boldsymbol{u}_C \rangle) = \boldsymbol{Q}\boldsymbol{X}, \tag{B 25}$$

or

$$\boldsymbol{X} = -\boldsymbol{Q}^T \dot{\boldsymbol{Q}} \cdot (\bar{\boldsymbol{w}} - \langle \boldsymbol{u}_C \rangle) + \boldsymbol{Q}^T \boldsymbol{X}^*, \tag{B 26}$$

where the superscript T indicates the transpose. This relation is general and must hold for any rotating frame, i.e. any $\boldsymbol{Q}$. Let us consider then the particular $\boldsymbol{Q}$ necessary to return to the laboratory frame countering the rotation $\boldsymbol{\Omega}$ of the rotating frame. In this frame $[\nabla^* \langle \boldsymbol{u}^*_C \rangle - (\nabla^* \langle \boldsymbol{u}^*_C \rangle)^T]$ must vanish and, since (Drew *et al.* 1979)

$$\nabla^* \langle \boldsymbol{u}^*_C \rangle = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \boldsymbol{Q}(\nabla \langle \boldsymbol{u}_C \rangle)\, \boldsymbol{Q}^T, \tag{B 27}$$

we find

$$0 = \nabla^* \langle u_C^* \rangle - (\nabla^* \langle u_C^* \rangle)^T$$
$$= \dot{\boldsymbol{Q}}\boldsymbol{Q}^T - \boldsymbol{Q}\dot{\boldsymbol{Q}}^T + \boldsymbol{Q}[\nabla \langle \boldsymbol{u}_C \rangle - (\nabla \langle \boldsymbol{u}_C \rangle)^T]\boldsymbol{Q}^T, \tag{B 28}$$

from which, using the orthogonality of $\boldsymbol{Q}$,

$$\dot{\boldsymbol{Q}}^T\boldsymbol{Q} - \boldsymbol{Q}^T\dot{\boldsymbol{Q}} = \nabla \langle \boldsymbol{u}_C \rangle - (\nabla \langle \boldsymbol{u}_C \rangle)^T. \tag{B 29}$$

Upon differentiation of the relation $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$, with $\boldsymbol{I}$ the identity tensor, it is however seen that $\dot{\boldsymbol{Q}}^T$ and $\boldsymbol{Q}$ anticommute so that the previous relation can also be written as

$$2\boldsymbol{Q}^T\dot{\boldsymbol{Q}} = -[\nabla \langle \boldsymbol{u}_C \rangle - (\nabla \langle \boldsymbol{u}_C \rangle)^T]. \tag{B 30}$$

Since in the new frame $\langle \boldsymbol{u}_C \rangle$ is irrotational, (5.5) is applicable so that $\boldsymbol{X}^*$ vanishes. Upon substitution of (B 30) into (B 26) one then finds

$$\boldsymbol{X} = -\tfrac{1}{2}(\bar{\boldsymbol{w}} - \langle \boldsymbol{u}_C \rangle) \times (\nabla \times \langle \boldsymbol{u}_C \rangle), \tag{B 31}$$

which is the result (5.9) quoted in the text.

Appendix C

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We derive here in a compact way a corrected form of the expression of Sangani & Didwania (1993*a*) for the dispersed-phase impulse equation and the 'particle' or disperse-phase stress tensor. This result may be obtained in many ways. An *ab initio* derivation is given in Bulthuis *et al.* (1994). Here we obtain it by starting from the averaged momentum equations of §4. The recovery of the result of Bulthuis *et al.* (1994) and Sangani & Didwania (1993*a*) in this way lends further support to the correctness of the equations found in this paper.

For simplicity, following Sangani & Didwania (1993*a*), we restrict our discussion to the case of massless particles and also ignore gravitational and collisional forces. Then the disperse- and continuous-phase momentum equations (4.11), (4.12) may be written

$$\rho_C \frac{\partial \beta_C \langle \boldsymbol{u}_C \rangle}{\partial t} + \rho_C \nabla \cdot (\beta_C \langle \boldsymbol{u}_C \boldsymbol{u}_C \rangle) + \beta_C \nabla \langle p_C \rangle = -\beta_D \boldsymbol{A}_D + \nabla \cdot (\beta_D \boldsymbol{T}_C), \tag{C 1}$$

$$\beta_D \nabla \langle p_C \rangle = \beta_D \boldsymbol{A}_D. \tag{C 2}$$

Adding and applying the transport theorem of (2.31) we write

$$\rho_C \beta_C \left\langle \frac{\partial \boldsymbol{u}_C}{\partial t} + \nabla \cdot (\boldsymbol{u}_C \boldsymbol{u}_C) \right\rangle = -\nabla \langle p_C \rangle + \nabla \cdot (\beta_D \boldsymbol{T}_C). \tag{C 3}$$

Let now ϕ_C be the exact microscopic velocity potential such that $\boldsymbol{u}_C = \nabla \phi_C$ and apply the relation (2.23) to find

$$\rho_C \beta_C \left\langle \frac{\partial \boldsymbol{u}_C}{\partial t} + \nabla \cdot (\boldsymbol{u}_C \boldsymbol{u}_C) \right\rangle = \rho_C \beta_C \left\langle \nabla \cdot \left(\frac{\partial \phi_C}{\partial t} \boldsymbol{I} + \boldsymbol{u}_C \boldsymbol{u}_C \right) \right\rangle$$
$$= \nabla \cdot \left(\rho_C \beta_C \left\langle \frac{\partial \phi_C}{\partial t} \boldsymbol{I} + \boldsymbol{u}_C \boldsymbol{u}_C \right\rangle \right)$$
$$- \rho_C \int \mathrm{d}^3 w \int_{|x-y|=a} \mathrm{d}S_y \, P(\boldsymbol{y}, \boldsymbol{w}, t) \boldsymbol{n} \cdot \left\langle \frac{\partial \phi_C}{\partial t} \boldsymbol{I} + \boldsymbol{u}_C \boldsymbol{u}_C \right\rangle_1 (\boldsymbol{x}, t \,|\, \boldsymbol{y}, \boldsymbol{w}). \tag{C 4}$$

We can convert the surface integral over the particles that touch the point $\boldsymbol{x}$ to an integral over the surface of the particle centred at $\boldsymbol{x}$ by using a Taylor expansion similar to that of (4.8). In this way we have

$$\rho_C \beta_C \left\langle \frac{\partial \boldsymbol{u}_C}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{u}_C \boldsymbol{u}_C) \right\rangle = \boldsymbol{\nabla}\cdot\left(\rho_C \beta_C \left\langle \frac{\partial \phi_C}{\partial t}\boldsymbol{I} + \boldsymbol{u}_C \boldsymbol{u}_C \right\rangle \right)$$
$$- \rho_C \int \mathrm{d}^3 w\, P(\boldsymbol{x}, \boldsymbol{w}, t) \int_{|\boldsymbol{x}-\boldsymbol{y}|=a} \mathrm{d}S_y\, \boldsymbol{n}\cdot \left\langle \frac{\partial \phi_C}{\partial t}\boldsymbol{I} + \boldsymbol{u}_C \boldsymbol{u}_C \right\rangle_1 (\boldsymbol{y}, t \,|\, \boldsymbol{x}, \boldsymbol{w})$$
$$+ \boldsymbol{\nabla}\cdot\left[\rho_C \int \mathrm{d}^3 w\, P(\boldsymbol{x}, \boldsymbol{w}, t) \int_{|\boldsymbol{x}-\boldsymbol{y}|=a} \mathrm{d}S_y\, \boldsymbol{s}\boldsymbol{n}\cdot \left\langle \frac{\partial \phi_C}{\partial t}\boldsymbol{I} + \boldsymbol{u}_C \boldsymbol{u}_C \right\rangle_1 (\boldsymbol{y}, t \,|\, \boldsymbol{x}, \boldsymbol{w}) \right]. \tag{C 5}$$

We now note that, if $\boldsymbol{J}$ denotes the Kelvin impulse of the generic particle centred at $\boldsymbol{x}$,

$$\boldsymbol{J} \equiv -\rho_C \int_{|\boldsymbol{x}-\boldsymbol{y}|=a} \mathrm{d}S_y\, \boldsymbol{n}\phi_C, \tag{C 6}$$

we have, in view of (2.38),

$$n\bar{\boldsymbol{J}} = -\rho_C \int \mathrm{d}^3 w\, P(\boldsymbol{x}, \boldsymbol{w}, t) \int_{|\boldsymbol{x}-\boldsymbol{y}|=a} \mathrm{d}S_y\, \boldsymbol{n} \left\langle \frac{\partial \phi_C}{\partial t} + \boldsymbol{w}\cdot\boldsymbol{u}_C \right\rangle_1 (\boldsymbol{y}, t \,|\, \boldsymbol{x}, \boldsymbol{w})$$
$$= -\rho_C \int \mathrm{d}^3 w\, P(\boldsymbol{x}, \boldsymbol{w}, t) \int_{|\boldsymbol{x}-\boldsymbol{y}|=a} \mathrm{d}S_y\, \boldsymbol{n}\cdot \left\langle \frac{\partial \phi_C}{\partial t}\boldsymbol{I} + \boldsymbol{u}_C \boldsymbol{u}_C \right\rangle_1 (\boldsymbol{y}, t \,|\, \boldsymbol{x}, \boldsymbol{w}), \tag{C 7}$$

so that we may also write

$$\rho_C \beta_C \left\langle \frac{\partial \boldsymbol{u}_C}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{u}_C \boldsymbol{u}_C) \right\rangle = \boldsymbol{\nabla}\cdot\left(\rho_C \beta_C \left\langle \frac{\partial \phi_C}{\partial t}\boldsymbol{I} + \boldsymbol{u}_C \boldsymbol{u}_C \right\rangle \right) + n\bar{\boldsymbol{J}}$$
$$+ \boldsymbol{\nabla}\cdot\left[\rho_C \int \mathrm{d}^3 w\, P(\boldsymbol{x}, \boldsymbol{w}, t) \int_{|\boldsymbol{x}-\boldsymbol{y}|=a} \mathrm{d}S_y\, \boldsymbol{s}\boldsymbol{n}\cdot \left\langle \frac{\partial \phi_C}{\partial t}\boldsymbol{I} + \boldsymbol{u}_C \boldsymbol{u}_C \right\rangle_1 (\boldsymbol{y}, t \,|\, \boldsymbol{x}, \boldsymbol{w}) \right]. \tag{C 8}$$

Upon using this relation to express the left-hand side of (C 3) and rearranging we then may write

$$n\bar{\boldsymbol{J}} = -\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^p, \tag{C 9}$$

where $\boldsymbol{\sigma}^p$ is the potential contribution to the particle stress given by

$$\boldsymbol{\sigma}^p = \rho_C \beta_C \langle \boldsymbol{u}_C \boldsymbol{u}_C \rangle + \rho_C \beta_C \left\langle \frac{\partial \phi_C}{\partial t} \right\rangle \boldsymbol{I} + \langle p_C \rangle \boldsymbol{I} - \beta_D \boldsymbol{T}_C$$
$$+ \rho_C \int \mathrm{d}^3 w\, P(\boldsymbol{x}, \boldsymbol{w}, t) \int_{|\boldsymbol{x}-\boldsymbol{y}|=a} \mathrm{d}S_y\, \boldsymbol{s}\boldsymbol{n}\cdot \left\langle \frac{\partial \phi_C}{\partial t}\boldsymbol{I} + \boldsymbol{u}_C \boldsymbol{u}_C \right\rangle_1 (\boldsymbol{y}, t \,|\, \boldsymbol{x}, \boldsymbol{w}). \tag{C 10}$$

Equation (C 9) may be recast in a different form by using (2.40) to express the left-hand side, to find

$$\frac{\partial n\bar{\boldsymbol{J}}}{\partial t} + \boldsymbol{\nabla}\cdot(n\bar{\boldsymbol{w}}\bar{\boldsymbol{J}}) = -\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}^k + \boldsymbol{\sigma}^p), \tag{C 11}$$

where $\boldsymbol{\sigma}^k$ is the kinetic part of the disperse-phase stress tensor

$$\boldsymbol{\sigma}^k = n(\overline{\boldsymbol{w}\boldsymbol{J}} - \bar{\boldsymbol{w}}\bar{\boldsymbol{J}}). \tag{C 12}$$

Equations (C 11) and (C 12) were first given by Biesheuvel & Gorissen (1990) who, however, did not derive an explicit representation for $\boldsymbol{\sigma}^p$.

The potential part $\boldsymbol{\sigma}^p$ of the stress tensor may be rewritten in a different form by noting that, from the Bernoulli integral, we have

$$\langle p_C \rangle = -\rho_C \left\langle \frac{\partial \phi_C}{\partial t} + \tfrac{1}{2} u_C^2 \right\rangle. \tag{C 13}$$

Hence, with the definition (4.10) of $\boldsymbol{T}_C$,

$$\begin{aligned} \beta_D \boldsymbol{T}_C - \beta_D \langle p_C \rangle \boldsymbol{I} &\equiv -\rho_C \int \mathrm{d}^3 w\, P(\boldsymbol{x}, \boldsymbol{w}, t) \int_{|\boldsymbol{x}-\boldsymbol{y}|=a} \mathrm{d}S_y\, \boldsymbol{sn} \langle p_C \rangle_1 (\boldsymbol{y}, t \,|\, \boldsymbol{x}, \boldsymbol{w}) \\ &= \rho_C \int \mathrm{d}^3 w\, P(\boldsymbol{x}, \boldsymbol{w}, t) \int_{|\boldsymbol{x}-\boldsymbol{y}|=a} \mathrm{d}S_y\, \boldsymbol{sn} \cdot \left\langle \frac{\partial \phi_C}{\partial t} \boldsymbol{I} + \tfrac{1}{2} u_C^2 \boldsymbol{I} \right\rangle_1 (\boldsymbol{y}, t \,|\, \boldsymbol{x}, \boldsymbol{w}). \end{aligned} \tag{C 14}$$

With this relation, (C 10) for the potential part of the particle stress finally becomes

$$\boldsymbol{\sigma}^p = \boldsymbol{M} - \tfrac{1}{2} \mathrm{Tr}[\boldsymbol{M}] \boldsymbol{I} + \rho_C \int \mathrm{d}^3 w\, P(\boldsymbol{x}, \boldsymbol{w}; t) \int_{|\boldsymbol{x}-\boldsymbol{y}|=a} \boldsymbol{sn} \cdot \langle \boldsymbol{u}_C \boldsymbol{u}_C - \tfrac{1}{2} u_C^2 \boldsymbol{I} \rangle_1 (\boldsymbol{y}, t \,|\, \boldsymbol{x}, \boldsymbol{w}), \tag{C 15}$$

where $\boldsymbol{M} = \rho_C \beta_C \langle \boldsymbol{u}_C \boldsymbol{u}_C \rangle$ is the average momentum flux of the continuous phase. After correction of some errors, this expression can be brought to coincide that of Sangani & Didwania (1993*a*). An alternative derivation and a complete discussion of this matter is presented in Bulthuis *et al.* (1994).

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